AN

ELEMENTARY TREATISE

ON THE

THEORY OF EQUATIONS

BY

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PREFACE.

In this treatise it is my aim to give the elements of Determinants and the Theory of Equations in a form suitable, both in amount and quality of matter, for use in the undergraduate courses in our colleges and universities. To this end I have endeavored to make the work in every part readily intelligible to the average student who has become proficient in algebra and the elements of trigonometry. All use of the calculus has purposely been avoided. While the presentation of the subject has necessarily been condensed to suit the requirements of college courses, great pains has been taken not to sacrifice clearness to brevity. It is a short treatise, but not a syllabus.

Part I treats of Determinants. The first two chapters give the fundamental theorems, with examples for illustration. The third chapter consists of applications and special forms of determinants, followed by a collection of carefully selected examples. These three chapters on determinants should serve as a helpful introduction to the study of this interesting class of functions.

Part II treats of the Theory of Equations proper. The principal elementary theorems concerning algebraic and numerical equations are deduced. After a brief introduction, giving definitions, etc., there follows a chapter on Complex Quantities, a subject which seems worthy of more space than is usually allotted to it in so elementary a treatise. This chapter, however, is given not so much for use in the chapters that follow, as with the hope that it may prove useful to the

student who pursues later in his course the study of the Theory of Functions. As all the theorems considered have become classic, no special references to authors consulted seem necessary in the body of the book. After Chapter IV I have followed quite closely Burnside and Panton, though in some places the general arrangement has been altered to make the necessary abridgments while securing clearness, and, wherever it seemed desirable, the method of proof has been changed. Almost every theorem is elucidated by the complete solution of one or more representative examples. I desire to call special attention to this feature of the book, which will surely commend itself alike to teacher and pupil. In Chapter XI I have striven to make the rather complicated process of the solution of numerical equations as simple as possible. It would defeat the object of this treatise were much space devoted to these methods, which are laborious and of no great practical value, but what is given is complete in itself. Horner's method is explained in detail.

The following works have been most helpful in the preparation of the treatise, Muir and Burnside and Panton in particular furnishing many examples: Baltzer, *Theorie und Anwendung der Determinanten*, 1881; Burnside and Panton, *Theory of Equations*, 1892; Carnoy, *Cours d'Algèbre Supérieure*, 1892; Hoüel, *Cours de Calcul Infinitésimal*, 1878; Klempt, *Lehrbuch zur Einführung in die Moderne Algebra*, 1880; Muir, *A Treatise on Determinants*, 1882. Todhunter's *Theory of Equations*, Chrystal's *Algebra*, Vol. I, Scott's *Theory of Determinants*, and that excellent little American work by Professor L. G. Weld (*A Short Course in the Theory of Determinants*) should also be mentioned; and the author has consulted with profit the well-known works of Serret, Petersen, Biermann, Matthiessen, and others.

The author gratefully acknowledges his indebtedness to Dr. D. E. Smith, of the State Normal School, at Brockport, N.Y., to Professor William H. Echols, of the University of Virginia, who have read the manuscript and made suggestive criticisms, and to Professor R. D. Bohannan, of the Ohio State University, and Dr. J. H. Gore, of the Columbian University, Washington, who have kindly read the revised proof sheets, and given many valuable suggestions, though he does not wish to hold them in the least responsible for the method followed in the treatment of the subject, nor for any errors that may have crept into the work.

SAMUEL M. BARTON.

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THEORY OF EQUATIONS.

PART I. — DETERMINANTS.

CHAPTER I.

THE ORIGIN, NOTATION, AND GENERAL DEFINITION OF DETERMINANTS.

As an introduction to the Theory of Equations, it seems proper that we should devote a few chapters to the discussion of the important class of functions known as determinants.

Historical Note. The first notion of *Determinants* we owe to Leibnitz, who in 1693 had observed the peculiarity of the expressions which arise from the solution of linear equations.

These functions were first called "determinants" by Cauchy, this name being adopted by him from the writings of Gauss, who had applied it to certain special classes of these functions; namely, the discriminants of binary and ternary quadratic forms. After Leibnitz no further advance in the subject was made until Cramer, in 1750, was led to the study of such functions in connection with the analysis of curves. During the latter part of the eighteenth century, the subject was farther enlarged by the labors of Bezout, Laplace, Vandermonde, and Lagrange. In the present century the first mathematicians who were prominent in developing this branch of mathematics were Gauss and Cauchy, and the subject was also studied by Binet in France and Wronski in Italy. We are indebted to Cauchy for the first formal treatise on the subject. A great impetus was given to the study of these functions by the writings of Jacobi in *Crelle's Journal* in 1841. Among more recent writers who have advanced the subject may be mentioned Hermite, Hesse, Joachimsthal, Cayley, Sylvester, and Salmon.

Text-books on Determinants were written by Spottiswoode (1851), Brioschi (1854), Baltzer (1857), Günter (1875), Dostor (1877), Baraniecki (1879), Scott (1880), Muir (1882), Weld (1883), and others.*

* The general text-books on Higher Algebra that devote a chapter or two to Determinants are numerous. Some of these are referred to in the author's Preface. Gordon's *Vorle*

1. Permutations. Let there be a group of elements

$$a,\ b,\ c,\ d,\ e, \cdots,$$

or

$$a_1,\ a_2,\ a_3,\ a_4,\ a_5, \cdots,$$

represented by different letters or the same letter affected with indices arranged in order of increasing magnitude. If we assemble these elements by placing them in any order, the group thus obtained is called a *permutation*.

It is proved in algebra that the number of permutations of the members of a group of n things is

$$1 \cdot 2 \cdot 3 \cdots n, \text{ or } n!^*$$

If the members of a group are arranged alphabetically, or, when represented by a single letter, if the indices of that letter occur in order of increasing magnitude, they are said to be written in the *natural order*.

In one, and in only one, of the permutations of the members of a group, the members are arranged in their natural order. In every other permutation the natural order is more or less deranged.

2. Any two members of a group arranged in their natural order constitute a *permanence*. Thus, the pairs

$$ab,\quad ac,\quad bc,\quad bd,\quad 12,\quad 13,\quad 23,$$

are permanences.

Any two members of a group arranged in an order which is the reverse of the natural order constitute an *inversion*. Thus the permutation

$$eadcfb$$

presents eight inversions,

$$ea,\quad ed,\quad ec,\quad eb,\quad dc,\quad db,\quad cb,\quad fb;$$

ungen über Invariantentheorie I. Bd. might be mentioned in this connection. For extended bibliographical notice, see Günther's *Lehrbuch der Determinanten-Theorie*, pp. 208 and 209, Muir's *Theory of Determinants*, and Scott's *Theory of Determinants*.

* The symbol $\underline{n|}$, read "factorial n," is also used to denote the product of the first n whole numbers, but in printed work $n!$ is the most convenient symbol.

the permutation $a_5a_4a_1a_3a_2$

presents seven inversions of subscripts,

$$51,\ 54,\ 53,\ 52,\ 43,\ 42,\ 32.$$

The permutations of the members of a group are divided into two classes, the *even* or *positive* permutations, and the *odd* or *negative* permutations. Even permutations are those which contain an even number of inversions. Odd permutations are those which contain an odd number of inversions.

The permutations

$$deabc,\quad 32541$$

are even (positive), because each contains an even number (six) of inversions; while the permutations

$$daebc,\quad 32451,$$

are odd (negative), because each contains an odd number (five) of inversions.

3. Theorem. *A permutation changes its class, from even to odd or from odd to even, when any two of its members are interchanged.*

Let $a, b, c, \cdots, q$ be the indices of the elements of a certain permutation. Now form the product

$$\begin{aligned} P = (b-a)(c-a)(d-a) & \ \ .\ \ .\ \ .\ \ .\ \ (q-a) \\ (c-b)(d-b) & \ \ .\ \ .\ \ .\ \ .\ \ (q-b) \\ (d-c) & \ \ .\ \ .\ \ .\ \ .\ \ (q-c) \\ .\ \ .\ \ .\ \ .\ \ .\ \ .\ \ .\ \ .\ \ . & \ \ .\ \ .\ \ .\ \ .\ \ .\ \ . \\ & \qquad\qquad (q-p) \end{aligned}$$

of the differences two and two of these indices, taken by subtracting each of them from all those which come after it in this permutation. To each inversion there will correspond a negative difference; therefore, P will be positive or negative according as the permutation belongs to the even or to the odd class.

This assumed, let g, k be the indices of two elements which

are to be interchanged; the product P, relative to the original permutation, can be put under the form

$$P = \overbrace{\pm 1(b-a)(c-a)\cdots}^{1} \times \overbrace{(g-a)(g-b)\cdots}^{2} \times \overbrace{(k-a)(k-b)\cdots}^{3} \times \overbrace{(k-g)}^{4},$$

the group (1) embracing all the factors not contained in the groups (2), (3), (4).

If we interchange g and k, the group of factors (1) undergoes no change; the groups (2) and (3) will only be interchanged, the one for the other; the factor (4) alone will change its sign, and, therefore, the permutation will change its class, which was to be proved.

The sign of the product P, which determines whether the permutation is even or odd, is called, for brevity, the *sign* of the permutation, and hence the name *positive* and *negative* is given to the *even* class and *odd* class respectively.

4. Theorem. *Of all possible permutations of the members of a group, one-half are even and one-half are odd.*

Suppose all the possible permutations written down. Now, let a new set of permutations be formed by fixing upon any two of the members and interchanging them in each permutation. The even permutations will thus be changed to odd, and the odd to even. That is, for every even permutation in the old set there is an odd one in the new, and *vice versa.* But, as is evident, the new set of permutations is the same as the old, only differently arranged. Hence in either set there are as many even as there are odd permutations, or one-half the permutations are even and one-half are odd.

5. Definition. We can now give our first definition of a determinant. A determinant of order n is the algebraic sum of the permutations of a product of n elements

$$a_1 b_2 c_3 \cdots k_{n-1} l_n$$

obtained either by the interchange of letters, or by the interchange of subscripts,

[We must of course remember that permutations of the *even* class have the + sign; those of the *odd* class, the − sign.]

The function $a_1b_2 - a_2b_1$ of the four quantities

$$\begin{matrix} a_1, & b_1, \\ a_2, & b_2, \end{matrix}$$

is obtained by assigning to a and b written in alphabetical order, the suffixes 1, 2, and 2, 1, corresponding to the two permutations of the numbers 1, 2 (the second term being minus, because 2, 1 is *odd*); and adding the two products so formed.

Similarly the function

$$a_1b_2c_3 - a_1b_3c_2 + a_2b_3c_1 - a_2b_1c_3 + a_3b_1c_2 - a_3b_2c_1 \qquad (1)$$

of the nine quantities

$$\begin{matrix} a_1, & b_1, & c_1, \\ a_2, & b_2, & c_2, \\ a_3, & b_3, & c_3, \end{matrix}$$

is obtained by adding algebraically all the products abc which can be formed by assigning to the letters (retained in their alphabetical order) suffixes corresponding to all the permutations of the numbers 1, 2, 3.

In like manner, we could form a similar function of the 4th order, of the sixteen quantities

$$\begin{matrix} a_1, & b_1, & c_1, & d_1, \\ a_2, & b_2, & c_2, & d_2, \\ a_3, & b_3, & c_3, & d_3, \\ a_4, & b_4, & c_4, & d_4. \end{matrix}$$

These functions are Determinants according to our first definition. In these functions, the quantities a_1, b_1, c_1, d_1, a_2, etc., are called *elements*, or *constituents*.

6. Second Definition. We see from the foregoing that a determinant embraces a square number of elements, and this leads us to a notation — a square array of the elements between two vertical lines, thus:

$$\Delta \equiv \begin{vmatrix} a_1 & b_1 & c_1 & \cdots & l_1 \\ a_2 & b_2 & c_2 & \cdots & l_2 \\ a_3 & b_3 & c_3 & \cdots & l_3 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ a_n & b_n & c_n & \cdots & l_n \end{vmatrix} \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

and we give as our second definition, embodying this notation,— a determinant of a system of n^2 elements, which are arranged in n rows of n elements each, or n columns of n elements each, is the algebraic sum of all possible products of n of these elements, no two of which belong to one row or to one column, the sign of any product being $+$, if the term is an even permutation; $-$, if the term is an odd permutation.

7. It follows at once from this definition that a general rule for the expansion of a determinant array is:

Write down all the products which can be formed by taking as factors one, and only one, element from each column and each row of the array. Of these products, the number of which is $n!$, one half involve the even permutations, and the other half involve the odd permutations of the subscripts $1, 2, 3, \cdots n$.

Now give to those products the *positive* sign, if the permutations of the subscripts are *even*; the *negative* sign, if the permutations are *odd*, and take their algebraic sum. The result is the expanded form of the determinant array. This method of expansion is, however, of little practical value.

8. Rule of Signs. The diagonal $a_1b_2c_3 \cdots l_n$ is called the *principal diagonal* of the determinant. The product $a_1b_2c_3 \cdots l_n$ of the letters in the principal diagonal always has the sign $+$, because in this permutation the letters as well as the suffixes occur in the natural order.

To determine the sign of any other product: first, put its letters in alphabetical order; then count the interchanges necessary to bring the subscripts into the order, $1, 2, 3, 4, \cdots$, of the subscripts in the principal diagonal. If they make an

even number, the term is affected with $+$; if they make an odd number, the term is affected with $-$.

A better rule is: *To determine the sign of any term, count its number of inversions, making the sign plus or minus according as that number is even or odd.*

EXAMPLES.

1. What sign is to be attached to the term $a_3b_7c_6d_5e_1f_4g_2$ in the determinant of the seventh order?

Here four interchanges put the subscript 1 first, then five interchanges put 2 in second place, then three put 4 in fourth place, then two put 5 in fifth place, and finally one interchange puts 6 between 5 and 7; hence in all there are fifteen interchanges, consequently the sign of the term is $-$. Or, more simply, the number of inversions is 15, an odd number, therefore the sign of the term is minus.

2. In a determinant of the fourth order, find the signs of the terms: $a_1b_4c_2d_3$; $a_1b_4c_3d_2$; $a_3b_1c_4d_2$; $a_4b_3c_2d_1$.

3. In a determinant of the fifth order, find the signs of the terms: $a_3b_4c_2d_5e_1$; $a_4b_2c_5d_1e_3$; $b_1a_2d_3c_5e_4$; $e_1a_3d_4b_5c_2$.

9. Determinants most frequently occur as the result of elimination from linear equations. For example, solving the two simultaneous linear equations,

$$a_1x + b_1y = m_1,$$

$$a_2x + b_2y = m_2,$$

we readily get

$$x = \frac{m_1b_2 - m_2b_1}{a_1b_2 - a_2b_1} \qquad (1)$$

and

$$y = \frac{a_1m_2 - a_2m_1}{a_1b_2 - a_2b_1} \qquad (2)$$

The common denominator of these two values of x and y is a function of the coefficients of x and y in the given simultaneous equations. This function

$$a_1b_2 - a_2b_1 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

is the determinant of the coefficients a_1, b_1, a_2, b_2, and is commonly expressed by the symbol

$$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

This symbol is called a *determinant array*, and the quantities a_1, b_1, a_2, b_2 are called *elements*. (Art. 5.)

The polynomial (3) is called the *expansion* of the determinant. Since each term of this expansion is the product of *two* elements, the determinant is said to be of the *second order*.

10. From our general theory, as given in Arts. 6, 7, and 8, we recognize the equation

$$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} \equiv a_1b_2 - a_2b_1$$

as an identity; but, in practice, we find more suitable rules for expanding determinants, and these we shall examine later.

The first member of the above identity is thus interpreted:

The determinant array of the second order must be understood to mean that the product of the elements on the diagonal passing from the lower left-hand corner to the upper right-hand corner of the array is to be subtracted from the product of the elements of the other diagonal.

Here and elsewhere we use the sign $\equiv$ to denote identity.

11. The numerators of the fractions in equations (1) and (2) of Art. 9 may also be written in the form of determinant arrays. Thus the values of x and y in the given simultaneous equations are

$$x = \frac{\begin{vmatrix} m_1 & b_1 \\ m_2 & b_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}}, \text{ and } y = \frac{\begin{vmatrix} a_1 & m_1 \\ a_2 & m_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}}.$$

EXAMPLES.

1. Expand $\begin{vmatrix} a & b \\ b & a \end{vmatrix}$. *Ans.* $a^2 - b^2$.

2. Evaluate $\begin{vmatrix} 1 & -2 \\ 4 & 3 \end{vmatrix}$. *Ans.* 11.

3. Evaluate $\begin{vmatrix} 0 & -1 \\ 9 & 0 \end{vmatrix}$.

4. Evaluate $\begin{vmatrix} 1 & 4 \\ 0 & 0 \end{vmatrix}$.

5. Evaluate $\begin{vmatrix} 100 & 50 \\ 50 & 25 \end{vmatrix}$.

6. Expand and reduce $\begin{vmatrix} \cos x & \sin x \\ \sin y & \cos y \end{vmatrix}$. *Ans.* $\cos(x+y)$.

7. Expand and reduce $\begin{vmatrix} -1 & \sin\alpha \\ \sin\alpha & -1 \end{vmatrix}$.

8. Expand and reduce $\begin{vmatrix} 1 & -\tan x \\ \tan x & 1 \end{vmatrix}$.

9. Expand and reduce $\begin{vmatrix} x+y & x-y \\ x+y & x-y \end{vmatrix}$.

Solve, by Arts. 9 and 11, the following simultaneous equations:

10. $6x + 5y = 46$, $10x + 3y = 66$.

11. $2x + 7y = 52$, $3x - 5y = 16$.

12. $2x - 7y = 8$, $4y - 9x = 19$.

13. $\dfrac{3}{x} + \dfrac{8}{y} = 3$, $\dfrac{15}{x} - \dfrac{4}{y} = 4$.

14. $ax - by = c, \quad cx + ay = b.$

12. Again, solving the three simultaneous linear equations,

$$a_1x + b_1y + c_1z = m_1,$$
$$a_2x + b_2y + c_2z = m_2,$$
$$a_3x + b_3y + c_3z = m_3,$$

we obtain

$$x = \frac{m_1b_2c_3 - m_1b_3c_2 + m_2b_3c_1 - m_2b_1c_3 + m_3b_1c_2 - m_3b_2c_1}{a_1b_2c_3 - a_1b_3c_2 + a_2b_3c_1 - a_2b_1c_3 + a_3b_1c_2 - a_3b_2c_1} \quad . \quad . \quad (1)$$

$$y = \frac{a_1m_2c_3 - a_1m_3c_2 + a_2m_3c_1 - a_2m_1c_3 + a_3m_1c_2 - a_3m_2c_1}{a_1b_2c_3 - a_1b_3c_2 + a_2b_3c_1 - a_2b_1c_3 + a_3b_1c_2 - a_3b_2c_1} \quad . \quad . \quad (2)$$

and

$$z = \frac{a_1b_2m_3 - a_1b_3m_2 + a_2b_3m_1 - a_2b_1m_3 + a_3b_1m_2 - a_3b_2m_1}{a_1b_2c_3 - a_1b_3c_2 + a_2b_3c_1 - a_2b_1c_3 + a_3b_1c_2 - a_3b_2c_1}. \quad . \quad . \quad (3)$$

The common denominator of these three fractions, which express the values of x, y, and z, is the determinant

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

The function

$$a_1b_2c_3 - a_1b_3c_2 + a_2b_3c_1 - a_2b_1c_3 + a_3b_1c_2 - a_3b_2c_1 \quad . \quad . \quad (5)$$

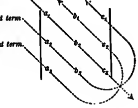
is called the expansion of the determinant (4), and since each term of this expansion is the product of *three* elements, the determinant is said to be of the *third order*.

Note. Such examples as the one given here are simply to show that determinants often occur as the result of elimination. The reader will learn in Chapter III that the process of elimination is much simplified by the use of determinants.

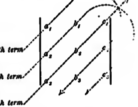
13. Since the determinant (4) is identically equal to the function (5), we have, arranging the terms of (5) in a convenient order,

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \equiv a_1b_2c_3 + a_2b_3c_1 + a_3b_1c_2 - a_3b_2c_1 - a_2b_1c_3 - a_1b_3c_2.$$

If a line be drawn through each triad of letters forming a term, we have the following diagrams, which furnish an excellent device for assisting the memory in expanding a determinant of the third order, viz., for the positive terms:



and for the negative terms:



In making practical use of these diagrams, it is customary to carry out the multiplication as each stroke is made. No similar diagram exists for a determinant of higher order than the third.

EXAMPLES.

Expand by this method the following determinants:

1. $\begin{vmatrix} x & y & z \\ v & w & u \\ t & r & s \end{vmatrix}$ 2. $\begin{vmatrix} a & -2a & b \\ 3b & -c & 4d \\ 2c & 3d & -4b \end{vmatrix}$

3. $\begin{vmatrix} x & -2z & -y^2 \\ -y & -2x & z^2 \\ -z & 2y & -x^2 \end{vmatrix}$ 4. $\begin{vmatrix} a & a & a \\ b & a & b \\ c & c & a \end{vmatrix}$

Evaluate the determinants,

5. $\begin{vmatrix} 4 & 5 & 2 \\ -1 & 2 & -3 \\ 6 & -4 & 5 \end{vmatrix}$ 6. $\begin{vmatrix} 2 & 0 & 3 \\ 1 & 3 & 5 \\ 2 & 6 & 10 \end{vmatrix}$

7. $\begin{vmatrix} 4 & -1 & -2 \\ 0 & 3 & 0 \\ 3 & -7 & 4 \end{vmatrix}$ 8. $\begin{vmatrix} 15 & 4 & -3 \\ 2 & 10 & 5 \\ 0 & 3 & 7 \end{vmatrix}$

9. $\begin{vmatrix} 9 & 8 & 7 \\ 6 & 5 & 4 \\ 3 & 2 & 1 \end{vmatrix}$ 10. $\begin{vmatrix} -1 & 0 & 4 \\ 2 & 3 & 5 \\ 4 & 6 & 0 \end{vmatrix}$

14. The numerators of the fractions in Art. 12 may also be written in the form of determinant arrays, and thus we have for the values of x, y, and z in the given simultaneous equations:

$$x = \frac{\begin{vmatrix} m_1 & b_1 & c_1 \\ m_2 & b_2 & c_2 \\ m_3 & b_3 & c_3 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}}, \quad y = \frac{\begin{vmatrix} a_1 & m_1 & c_1 \\ a_2 & m_2 & c_2 \\ a_3 & m_3 & c_3 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}},$$

and

$$z = \frac{\begin{vmatrix} a_1 & b_1 & m_1 \\ a_2 & b_2 & m_2 \\ a_3 & b_3 & m_3 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}}.$$

NOTE. It may be observed that the numerator of the fraction expressing the value of x may be formed from the denominator of the same fraction by replacing a_1, a_2, a_3, the coefficients of x, by the absolute terms m_1, m_2, m_3, respectively. Similarly for y and z.

15. The solution of the four simultaneous equations

$$\begin{aligned} a_1x + b_1y + c_1z + d_1w &= m_1 \\ a_2x + b_2y + c_2z + d_2w &= m_2 \\ a_3x + b_3y + c_3z + d_3w &= m_3 \\ a_4x + b_4y + c_4z + d_4w &= m_4 \end{aligned} \tag{1}$$

would show that the values of x, y, z, and w are expressed by fractions having a common denominator which is a function of the sixteen coefficients a_1, b_1, c_1, d_1, a_2, $\cdots$, etc. This function is a determinant of the fourth order. We have the following identical equation:

$$\begin{vmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \\ a_4 & b_4 & c_4 & d_4 \end{vmatrix} \begin{aligned} &\equiv a_1b_2c_3d_4 - a_1b_2c_4d_3 - a_1b_3c_2d_4 + a_1b_4c_2d_3 \\ &+ a_1b_3c_4d_2 - a_1b_4c_3d_2 - a_2b_1c_3d_4 + a_2b_1c_4d_3 \\ &+ a_3b_1c_2d_4 - a_4b_1c_2d_3 - a_3b_1c_4d_2 + a_4b_1c_3d_2 \\ &+ a_2b_3c_1d_4 - a_2b_4c_1d_3 - a_3b_2c_1d_4 + a_4b_2c_1d_3 \\ &+ a_3b_4c_1d_2 - a_4b_3c_1d_2 - a_2b_3c_4d_1 + a_2b_4c_3d_1 \\ &+ a_3b_2c_4d_1 - a_4b_2c_3d_1 - a_3b_4c_2d_1 + a_4b_3c_2d_1 \; . \end{aligned} \tag{2}$$

The solution of five simultaneous linear equations involving five unknown quantities would give rise to a determinant of the fifth order, the expanded form of which contains $120 (= 5!)$ terms.

$$\begin{vmatrix} a_1 & b_1 & c_1 & d_1 & e_1 \\ a_2 & b_2 & c_2 & d_2 & e_2 \\ a_3 & b_3 & c_3 & d_3 & e_3 \\ a_4 & b_4 & c_4 & d_4 & e_4 \\ a_5 & b_5 & c_5 & d_5 & e_5 \end{vmatrix} \equiv a_1b_2c_3d_4e_5 \pm \text{etc.} \quad . \quad . \quad (3)$$

Similarly, the solution of n simultaneous linear equations involving n unknown quantities would give rise to a determinant of the nth order, the expanded form of which contains $n!$ terms. The determinant array may be written thus:

$$\begin{vmatrix} a_1 & b_1 & c_1 & \cdots & l_1 \\ a_2 & b_2 & c_2 & \cdots & l_2 \\ a_3 & b_3 & c_3 & \cdots & l_3 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ a_n & b_n & c_n & \cdots & l_n \end{vmatrix} \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

16. It is evident from the foregoing that the *determinant of the nth order involves n^2 elements*, which agrees with Art. 6.

The horizontal ranks of elements are called *rows* of the determinant, and the vertical ranks are called *columns*. The rows are numbered from the top row downward, and the columns from the left-hand column to the right. A *line* is either a row or a column.

In any determinant, the diagonal from the upper left-hand corner to the lower right-hand corner is called the *principal diagonal*, as we have had occasion to remark, and the other is called the *secondary diagonal*. The terms of the expansion, which are the products of the elements on these diagonals, are called respectively the *principal term* and the *secondary term*. Thus in the determinant (4), of the preceding article, $a_1b_2c_3 \cdots l_n$ is the principal term, $a_nb_{n-1}c_{n-2} \cdots l_1$ is the secondary term.

17. Another notation for the determinant of the nth order is the following:

$$\begin{vmatrix} a_1' & a_1'' & a_1''' & \cdots & a_1^{(n)} \\ a_2' & a_2'' & a_2''' & \cdots & a_2^{(n)} \\ a_3' & a_3'' & a_3''' & \cdots & a_3^{(n)} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ a_n' & a_n'' & a_n''' & \cdots & a_n^{(n)} \end{vmatrix} \quad . \quad . \quad . \quad . \quad . \quad (1)$$

in which the number of the *row* is indicated by the *subscript*, and the number of the *column* by the *superscript*.

Another notation, and one that is very much used, is the following:

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn} \end{vmatrix} \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Here the number of the *row* is indicated by the *first* of the two subscripts, and the number of the *column* by the *second*. Thus, the element a_{35} of the above array is in the third row, and the fifth column.

There are several simpler methods for writing determinants, when it is perfectly well understood what the elements of the determinants are.

Thus, if Δ denotes the determinant (4) of Art. 15, it may, for brevity, be represented in the following ways:

$$\Delta \equiv (a_1 b_2 c_3 \cdots l_n) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

$$\Delta \equiv |a_1 b_2 c_3 \cdots l_n| \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

that is, simply by placing the principal term within brackets. The notation $\Sigma \pm a_1 b_2 c_3 \cdots l_n$ is also used to represent Δ; this expressing its constitution as consisting of the sum of a number of terms (with their proper signs) formed by taking all possible permutations of the n suffixes. With this notation determinant (2) of this article would be expressed by

$$\Sigma \pm a_{11} a_{22} a_{33} \cdots a_{nn} \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

EXAMPLES.

1. Expand $\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$

Ans. $x_1y_2 + x_2y_3 + x_3y_1 - x_1y_3 - x_2y_1 - x_3y_2.$

2. Evaluate $\begin{vmatrix} 2 & -1 & -1 \\ -3 & -2 & 2 \\ 2 & 0 & 1 \end{vmatrix}$ *Ans.* $-15.$

3. Evaluate $\begin{vmatrix} 3 & 2 & 4 \\ 7 & 6 & 1 \\ 5 & 3 & 8 \end{vmatrix}$

4. Evaluate $\begin{vmatrix} 2 & -1 & 4 \\ 6 & 5 & 0 \\ -3 & 4 & 2 \end{vmatrix}$

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Ans. $20 + \cancel{3} + 96 - (-60) - 0 - (-12) = \cancel{191}.$

5. Expand $\begin{vmatrix} 1 & a & b \\ a & 1 & c \\ b & c & 1 \end{vmatrix}$

In the following examples express the values of x, y, and z in the notation of determinants, as in Art. 14, and then evaluate these determinants by the method of Art. 13.

6. Solve the simultaneous equations,

$$x + y - z = 1,$$
$$8x + 3y - 6z = 1,$$
$$-4x - y + 3z = 1.$$

Here, by Art. 14,

$$x=\frac{\begin{vmatrix} 1 & 1 & -1 \\ 1 & 3 & -6 \\ 1 & -1 & 3 \end{vmatrix}}{\begin{vmatrix} 1 & 1 & -1 \\ 8 & 3 & -6 \\ -4 & -1 & 3 \end{vmatrix}},\quad y=\frac{\begin{vmatrix} 1 & 1 & -1 \\ 8 & 1 & -6 \\ -4 & 1 & 3 \end{vmatrix}}{\begin{vmatrix} 1 & 1 & -1 \\ 8 & 3 & -6 \\ -4 & -1 & 3 \end{vmatrix}},\quad z=\frac{\begin{vmatrix} 1 & 1 & 1 \\ 8 & 3 & 1 \\ -4 & -1 & 1 \end{vmatrix}}{\begin{vmatrix} 1 & 1 & -1 \\ 8 & 3 & -6 \\ -4 & -1 & 3 \end{vmatrix}},$$

or $$x=\frac{-2}{-1}=2,\quad y=\frac{-3}{-1}=3,\quad z=\frac{-4}{-1}=4.$$

7. Solve the simultaneous equations,

$$\begin{aligned} 3x+2y-4z&=15, \\ 5x-3y+2z&=28, \\ -x+3y+4z&=24. \end{aligned}$$

Ans. $x=7$, $y=5$, $z=4$.

8. Solve the simultaneous equations,

$$\begin{aligned} 4x-3y+2z&=9, \\ 2x+5y-3z&=4, \\ 6y-2z+5x&=18. \end{aligned}$$

Ans. $x=2$, $y=3$, $z=5$.

9. Solve the simultaneous equations,

$$\begin{aligned} 3x+2y+z&=23, \\ 5x+2y+4z&=46, \\ 10x+5y+4z&=75. \end{aligned}$$

10. Solve the simultaneous equations,

$$\begin{aligned} 2x-7y+4z&=0, \\ 3x-3y+z&=0, \\ 9x+5y+3z&=28. \end{aligned}$$

CHAPTER II.

PROPERTIES OF DETERMINANTS.

From our definitions of a determinant, as given in Articles 5 and 6, we readily deduce the following important theorems:

18. THEOREM. *The value of a determinant is not changed by substituting the columns for corresponding rows and the rows for corresponding columns; that is,*

$$\begin{vmatrix} a_1 & b_1 & c_1 & \cdots & l_1 \\ a_2 & b_2 & c_2 & \cdots & l_2 \\ a_3 & b_3 & c_3 & \cdots & l_3 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ a_n & b_n & c_n & \cdots & l_n \end{vmatrix} = \begin{vmatrix} a_1 & a_2 & a_3 & \cdots & a_n \\ b_1 & b_2 & b_3 & \cdots & b_n \\ c_1 & c_2 & c_3 & \cdots & c_n \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ l_1 & l_2 & l_3 & \cdots & l_n \end{vmatrix}$$

For, the two determinants having the same principal term, they will be identical on account of the way in which all the other terms are deduced. (Art. 8.)

It follows that any theorem true in regard to the rows of a determinant is also true in regard to the columns, and *vice versa.*

19. THEOREM. *Interchanging any two rows (or columns) of a determinant, simply changes the sign of the determinant; that is,*

$$\begin{vmatrix} a_1 & b_1 & c_1 & \cdots & l_1 \\ a_2 & b_2 & c_2 & \cdots & l_2 \\ a_3 & b_3 & c_3 & \cdots & l_3 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ a_n & b_n & c_n & \cdots & l_n \end{vmatrix} = - \begin{vmatrix} a_2 & b_2 & c_2 & \cdots & l_2 \\ a_1 & b_1 & c_1 & \cdots & l_1 \\ a_3 & b_3 & c_3 & \cdots & l_3 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ a_n & b_n & c_n & \cdots & l_n \end{vmatrix} = - \begin{vmatrix} b_1 & a_1 & c_1 & \cdots & l_1 \\ b_2 & a_2 & c_2 & \cdots & l_2 \\ b_3 & a_3 & c_3 & \cdots & l_3 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ b_n & a_n & c_n & \cdots & l_n \end{vmatrix}$$

For this modification amounts to changing the index 1 with the index 2, or the letter a with the letter b in the different terms of the determinant, and we know that in this case the corresponding permutation changes its sign, and hence all the terms of the last two determinants will have the same absolute value as that of the first, but their signs will be different.

20. THEOREM. *If two rows (or columns) of a determinant are identical, the determinant is equal to zero;* thus

$$\Delta \equiv \begin{vmatrix} a_1 & b_1 & c_1 & \cdots & l_1 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ a_r & b_r & c_r & \cdots & l_r \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ a_r & b_r & c_r & \cdots & l_r \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ a_n & b_n & c_n & \cdots & l_n \end{vmatrix} = 0.$$

For, by interchanging the two identical rows, we obtain

$$\Delta = -\Delta,$$

$$\therefore 2\Delta = 0.$$

Whence $\Delta = 0.$

21. THEOREM. *If each element in any line be multiplied by the same factor, the determinant is multiplied by that factor:* thus:

$$\begin{vmatrix} ma_1 & b_1 & c_1 & \cdots & l_1 \\ ma_2 & b_2 & c_2 & \cdots & l_2 \\ ma_3 & b_3 & c_3 & \cdots & l_3 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ ma_n & b_n & c_n & \cdots & l_n \end{vmatrix} = m \begin{vmatrix} a_1 & b_1 & c_1 & \cdots & l_1 \\ a_2 & b_2 & c_2 & \cdots & l_2 \\ a_3 & b_3 & c_3 & \cdots & l_3 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ a_n & b_n & c_n & \cdots & l_n \end{vmatrix}$$

For every term of the determinant must contain one, and only one, element from any row or any column.

Cor. I. If the elements in any line are the same multiple of the corresponding elements of any other parallel line, the determinant vanishes.

$$\begin{vmatrix} a_1 & ma_1 & a_2 & a_3 \\ b_1 & mb_1 & b_2 & b_3 \\ c_1 & mc_1 & c_2 & c_3 \\ d_1 & md_1 & d_2 & d_3 \end{vmatrix} = m \begin{vmatrix} a_1 & a_1 & a_2 & a_3 \\ b_1 & b_1 & b_2 & b_3 \\ c_1 & c_1 & c_2 & c_3 \\ d_1 & d_1 & d_2 & d_3 \end{vmatrix} = 0.$$

Cor. II. If the signs of each element in any line be changed, the sign of the determinant is changed. For this is equivalent to multiplying by the factor -1.

EXAMPLES.

1. Show that the following determinant vanishes:

$$\begin{vmatrix} 4 & 3 & 2 & 1 \\ 8 & 8 & 7 & 2 \\ 16 & 2 & 8 & 4 \\ 12 & 6 & 3 & 3 \end{vmatrix}$$

When the elements of the first column are divided by 4, they become identical with those of the last column.

2. Prove the following identity:

$$\begin{vmatrix} 2 & 6 & 10 & 2 \\ 3 & 6 & 15 & 3 \\ 2 & 4 & 4 & 3 \\ 5 & 2 & 2 & 1 \end{vmatrix} \equiv 6 \begin{vmatrix} 1 & 3 & 5 & 1 \\ 1 & 2 & 5 & 1 \\ 2 & 4 & 4 & 3 \\ 5 & 2 & 2 & 1 \end{vmatrix}$$

3. Show that the following determinant vanishes:

$$\begin{vmatrix} 2 & 0 & 4 & 6 & 1 \\ 1 & 3 & 0 & 2 & 7 \\ 2 & 4 & 1 & 3 & 2 \\ 4 & 8 & 2 & 6 & 4 \\ 9 & 0 & 5 & 3 & 7 \end{vmatrix}$$

4. Prove the identity:

$$\begin{vmatrix} bc & a & a^2 \\ ca & b & b^2 \\ ab & c & c^2 \end{vmatrix} = \begin{vmatrix} 1 & a^2 & a^3 \\ 1 & b^2 & b^3 \\ 1 & c^2 & c^3 \end{vmatrix}$$

Represent the first determinant by Δ, and multiply the rows by a, b, c, respectively. We have then

$$abc\,\Delta = \begin{vmatrix} abc & a^2 & a^3 \\ abc & b^2 & b^3 \\ abc & c^2 & c^3 \end{vmatrix};$$

and, dividing the first column by abc, the result follows.

5. Prove the identity

$$\begin{vmatrix} \beta\gamma\delta & \alpha & \alpha^2 & \alpha^3 \\ \gamma\delta\alpha & \beta & \beta^2 & \beta^3 \\ \delta\alpha\beta & \gamma & \gamma^2 & \gamma^3 \\ \alpha\beta\gamma & \delta & \delta^2 & \delta^3 \end{vmatrix} \equiv \begin{vmatrix} 1 & \alpha^2 & \alpha^3 & \alpha^4 \\ 1 & \beta^2 & \beta^3 & \beta^4 \\ 1 & \gamma^2 & \gamma^3 & \gamma^4 \\ 1 & \delta^2 & \delta^3 & \delta^4 \end{vmatrix}$$

6. Prove

$$\begin{vmatrix} 2 & 1 & -7 \\ -4 & -3 & 8 \\ 6 & 5 & -9 \end{vmatrix} \equiv 2 \begin{vmatrix} 1 & 1 & 7 \\ 2 & 3 & 8 \\ 3 & 5 & 9 \end{vmatrix}$$

7. Prove the identity:

$$\begin{vmatrix} 1 & 1 & 1 \\ \alpha & \beta & \gamma \\ \alpha^2 & \beta^2 & \gamma^2 \end{vmatrix} \equiv (\beta-\gamma)(\gamma-\alpha)(\alpha-\beta).$$

Since if β were equal to γ, two columns would become identical, $\beta-\gamma$ must be a factor in the determinant. Similarly, $\gamma-\alpha$ and $\alpha-\beta$ must be factors in it. Hence the product of the three differences can differ by a numerical factor only from the value of the determinant, since both functions are of

the third degree in α, β, γ; and by comparing the term $\beta\gamma^2$ we observe that this factor is $+1$.

NOTE. If the student is not familiar with the application of the so-called "Remainder Theorem," he might find it to his advantage at this point to read Art. 82.

Examples 7 and 8 belong to a class of functions called "alternating," and these particular determinants are known as *simple alternants*. The following definitions may assist the student (see Crystal's *Algebra*, Vol. I, Chap. IV).

An integral function is said to be symmetrical (with respect to all its variables) when the interchange of any pair whatever of its variables would leave its value unaltered. For example, $yz+zx+xy$ is a symmetrical function of x, y, z.

An integral function is called "alternating," when the interchange of any pair whatever of its variables changes the sign only of the function. An example is $(\beta-\gamma)(\gamma-\alpha)(\alpha-\beta)$, or the determinant of Ex. 7.

Now $x^2y+y^2z+z^2x$ is an *asymmetrical* function; that is, it is not a symmetrical function of x, y, z, for the three interchanges x with y, x with z, y with z, give respectively

$$y^2x+x^2z+z^2y,$$
$$z^2y+y^2x+x^2z,$$
$$x^2z+z^2y+y^2x,$$

and, though these are all equal to each other, no one of them is equal to the original function. We observe from this instance that asymmetrical functions have a property, which symmetrical functions have not, of assuming different values when the variables are interchanged: thus $x^2y+y^2z+z^2x$ is susceptible of two different values under this treatment, and is called a two-valued function. The study of algebra from this point of view has developed into a beautiful branch of modern algebra, known as *the theory of substitutions* (or *the theory of groups*).*

8. Prove similarly the identity

$$\begin{vmatrix} 1 & 1 & 1 & 1 \\ \alpha & \beta & \gamma & \delta \\ \alpha^2 & \beta^2 & \gamma^2 & \delta^2 \\ \alpha^3 & \beta^3 & \gamma^3 & \delta^3 \end{vmatrix} \equiv -(\beta-\gamma)(\alpha-\delta)(\gamma-\alpha)(\beta-\delta)(\alpha-\beta)(\gamma-\delta).$$

* See Netto's *Substitutionstheorie*, Serret's *Cours D'Algèbre Supérieure*, Petersen's *Algebraische Gleichungen*.

9. Reduce the following determinant to one in which the first row shall consist of units:

$$\Delta \equiv \begin{vmatrix} 2 & 4 & 10 & 5 \\ 0 & 1 & 4 & 3 \\ 7 & 2 & 5 & 5 \\ 3 & 0 & 1 & 4 \end{vmatrix}$$

Since 20 is the least common multiple of 2, 4, 10, 5, it is sufficient to multiply the columns in order by 10, 5, 2, 4; we thus obtain

$$\Delta \equiv \frac{1}{2 \cdot 4 \cdot 10 \cdot 5} \begin{vmatrix} 20 & 20 & 20 & 20 \\ 0 & 5 & 8 & 12 \\ 70 & 10 & 10 & 20 \\ 30 & 0 & 2 & 16 \end{vmatrix}$$

Taking out the multiplier 20 from the first row, 10 from the third row, and 2 from the fourth row, we get finally

$$\Delta \equiv \begin{vmatrix} 1 & 1 & 1 & 1 \\ 0 & 5 & 8 & 12 \\ 7 & 1 & 1 & 2 \\ 15 & 0 & 1 & 8 \end{vmatrix}$$

10. Reduce the following determinant to one in which the first column shall consist of units:

$$\Delta \equiv \begin{vmatrix} 2 & 0 & 2 & 1 \\ 3 & 4 & 4 & 0 \\ 6 & 6 & 7 & 6 \\ 8 & 4 & 4 & 5 \end{vmatrix}$$

22. Determinant Minors. It is evident from the notation, in a square array, of a determinant of the nth order, that the suppression of p rows and of p columns leaves a square con-

taining no more than $n-p$ rows and $n-p$ columns. We thus obtain a series of determinants of lower order, which we call *minors* of the primitive determinant. For example, in

$$\Delta \equiv \begin{vmatrix} a_1 & b_1 & c_1 & \cdots & k_1 & l_1 \\ a_2 & b_2 & c_2 & \cdots & k_2 & l_2 \\ a_3 & b_3 & c_3 & \cdots & k_3 & l_3 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ a_n & b_n & c_n & \cdots & k_n & l_n \end{vmatrix},$$

let us suppress the row and the column which contain the element a_1; it will become the determinant of the $(n-1)$th order,

$$A_1 \equiv \begin{vmatrix} b_2 & c_2 & \cdots & k_2 & l_2 \\ b_3 & c_3 & \cdots & k_3 & l_3 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ b_n & c_n & \cdots & k_n & l_n \end{vmatrix} = \Sigma \pm b_2 c_3 \cdots l_n,$$

which is called the *first minor* of Δ with respect to the element a_1.

As we can repeat this operation on each element, a determinant of the nth order has as many *first minors* as it contains elements.

We designate these generally by the large letters $A, B, C, \cdots$, written with the same index as the corresponding element. These, arranged in the order of the elements, form the following table:

$$\begin{vmatrix} A_1 & B_1 & C_1 & \cdots & K_1 & L_1 \\ A_2 & B_2 & C_2 & \cdots & K_2 & L_2 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ A_n & B_n & C_n & \cdots & K_n & L_n \end{vmatrix}$$

When we suppress any two rows and two columns, the remaining determinant of the $(n-2)$th order is called the *second minor* of the original determinant.

By omitting, for example, the first two rows and the first two columns, we have the determinant

$$\begin{vmatrix} c_3 & d_3 & \cdots & k_3 & l_3 \\ c_4 & d_4 & \cdots & k_4 & l_4 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ c_n & d_n & \cdots & k_n & l_n \end{vmatrix};$$

$= \Delta_{a_1}(n-2)^{\text{th}}$ order

the rows and the columns suppressed have in common the elements of the determinant

$$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix},$$

and the determinant of the $(n-2)$th order which precedes is the second minor of Δ with respect to a determinant of the second order.

In general, by the suppression of p rows and of p columns, we get a determinant of the $(n-p)$th order which we call the pth minor of Δ corresponding to a determinant of the pth order formed by the elements common to the rows and columns suppressed. The minor thus formed is said to be *complementary* to the determinant formed by elements common to the suppressed rows and columns.

23. *Development of a determinant according to the elements of a row and of a column.*

Take the determinant of nth order.

$$\Delta \equiv \begin{vmatrix} a_1 & b_1 & c_1 & \cdots & k_1 & l_1 \\ a_2 & b_2 & c_2 & \cdots & k_2 & l_2 \\ a_3 & b_3 & c_3 & \cdots & k_3 & l_3 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ a_n & b_n & c_n & \cdots & k_n & l_n \end{vmatrix} \equiv \Sigma \pm a_1 b_2 c_3 \cdots k_{n-1} l_n.$$

The different terms of Δ which contain the element a_1 are obtained by forming all the possible permutations of the other

and multiplying each

elements, which gives the determinant of $(n-1)$th order, $\Sigma \pm b_2c_3 \cdots k_{n-1}l_n$, resulting from the suppression of the first row and the first column. The determinant Δ will contain then, first, a series of terms having a_1 as a factor represented by

$$a_1\Sigma \pm b_2c_3d_4 \cdots l_n.$$

Interchanging a and b, this becomes

$$-b_1\Sigma \pm a_2c_3d_4 \cdots l_n,$$

for all the terms containing b_1; the sign $-$ is caused by the permutation of a and b. The sum Σ designates the determinant obtained by omitting the first row and the second column.

By changing b into c, we get, similarly, the expression

$$c_1\Sigma \pm a_2b_3d_4 \cdots l_n,$$

which would represent the series of terms having c_1 as a factor; the sum Σ designating the determinant arising from the suppression of the first row and of the third column, and so on.

All the terms containing l_1 would be represented by

$$(-1)^{n-1}l_1\Sigma \pm a_2b_3c_4 \cdots k_n.$$

Now, by definition, the sums which accompany the elements a_1, $-b_1$, $+c_1$, $\cdots (-1)^{n-1}l_1$ are the *first minors* with respect to these elements. Therefore, we have the following formula:

$$\Delta = a_1A_1 \pm b_1B_1 + c_1C_1 + \cdots \pm l_1L_1, \quad (-1)^{n-1} l_1L_1 \quad . \quad . \quad . \quad (1)$$

with the condition that, according to the composition of the determinant, we must attribute to the minors signs alternately positive and negative. thereby modifying our def. of minors

The number of terms in the second member of this equality is evidently

$$n(1 \cdot 2 \cdot 3 \cdots n-1) = n!,$$

the same as the number of terms in the determinant.

Following the same reasoning, and interchanging successively the indices two and two, we arrive at the similar relation

$$\Delta = a_1A_1 \mp a_2A_2 + a_3A_3 \mp \cdots \pm a_nA_n, \quad . \quad . \quad . \quad (2)$$

where we must give to the minors the signs alternately $+$ and $-$. This formula gives the development of Δ according to the elements of the first column.

It is evident that there exists a similar development for each row and for each column. Finally, to fix the sign of the minors in each formula, we move the row, or the column that we are considering, to the first place by the interchanges of the rows or columns, in observing that the determinant changes only its sign for an odd number of interchanges, while it preserves the same sign for an even number. Thus,

$$\Delta = a_2A_2 + b_2B_2 + c_2C_2 + \cdots + l_2L_2 \quad . \quad . \quad . \quad (3)$$

$$\Delta = a_3A_3 + b_3B_3 + c_3C_3 + \cdots + l_3L_3; \quad . \quad . \quad . \quad (4)$$

in formula (3) we would alternate the signs commencing with the sign $-$ for A_2; for, to lead the second row to the first place, one interchange of two rows suffices; in formula (4) it is necessary to commence with the sign $+$ for A_3, since two interchanges of rows are required to lead the third row to the first place; and so on.

24. Let us apply these principles to a determinant of the third order:

$$\Delta \equiv \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

Expressing the minors with their proper signs, we have

$$\Delta \equiv a_1A_1 - b_1B_1 + c_1C_1 \equiv -a_2A_2 + b_2B_2 - c_2C_2 \equiv a_3A_3 - b_3B_3 + c_3C_3,$$

$$\Delta \equiv a_1A_1 - a_2A_2 + a_3A_3 \equiv -b_1B_1 + b_2B_2 - b_3B_3 \equiv c_1C_1 - c_2C_2 + c_3C_3.$$

or, replacing the minors by their values,

$$
\begin{aligned}
\Delta \equiv\ & a_1\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - b_1\begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} + c_1\begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix} \\
\equiv -& a_2\begin{vmatrix} b_1 & c_1 \\ b_3 & c_3 \end{vmatrix} + b_2\begin{vmatrix} a_1 & c_1 \\ a_3 & c_3 \end{vmatrix} - c_2\begin{vmatrix} a_1 & b_1 \\ a_3 & b_3 \end{vmatrix} \\
\equiv\ & a_3\begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix} - b_3\begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix} + c_3\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} \\
\equiv\ & a_1\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - a_2\begin{vmatrix} b_1 & c_1 \\ b_3 & c_3 \end{vmatrix} + a_3\begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix} \\
\equiv -& b_1\begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} + b_2\begin{vmatrix} a_1 & c_1 \\ a_3 & c_3 \end{vmatrix} - b_3\begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix} \\
\equiv\ & c_1\begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix} - c_2\begin{vmatrix} a_1 & b_1 \\ a_3 & b_3 \end{vmatrix} + c_3\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}
\end{aligned}
$$

By virtue of what precedes, we can operate as follows to ascertain the sign of a minor with respect to any element. For example, let it be proposed to find the sign of the minor of d_4 in a determinant of the nth order. Proceeding on the first row from a_1, alternating the signs until we get to the column of the d elements, we reach d_1 with the sign $-$; we descend then the column of the d elements, changing the sign each time that we cross a row until we arrive, in this manner, at d_4 with the sign $+$; therefore the minor of d_4 ought to be affected with the positive sign.

Again, let the determinant be represented by

$$
\Delta \equiv \begin{vmatrix}
a_{11} & a_{12} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & a_{1n} \\
a_{21} & a_{22} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & a_{2n} \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
a_{k1} & a_{k2} & \cdot & \cdot & \cdot & a_{kk} & \cdot & \cdot & \cdot & \cdot & a_{kn} \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
a_{n1} & a_{n2} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & a_{nn}
\end{vmatrix}
$$

and let us seek the sign of the minor relative to the element a_{kl}. To this end, we must by the interchange of rows and columns lead this element to the first place. By $l-1$ interchanges of two consecutive columns, the element a_{kl} will occupy the first position in the kth horizontal line; then by $k-1$ interchanges of rows, this element will take the first place in the first row. All these operations amount to multiplying the determinant by $(-1)^{k+l-2}$ or $(-1)^{k+l}$. The sign of the minor A_{kl} will therefore be positive if the sum of the indices of the element a_{kl} is even, and negative in the contrary case. It is useful to observe that the first minors of the elements of the diagonal are all positive.

The preceding developments lead to important consequences which we shall now give, for brevity making use of simple determinants in illustration.

25. Theorem. *When all the elements of a row or of a column are zero, the determinant is equal to zero.* Thus

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ 0 & 0 & 0 \end{vmatrix} = 0, \qquad \begin{vmatrix} 0 & b_1 & c_1 \\ 0 & b_2 & c_2 \\ 0 & b_3 & c_3 \end{vmatrix} = 0,$$

for all the terms of the development according to the elements of these lines become zero by the presence of the factor zero.

26. Theorem. *When each of the elements of a row (or column) is zero except one of them, the order of the determinant is lowered by unity.*

We have, for example,

$$\begin{vmatrix} a_1 & 0 & 0 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = a_1A_1 + 0 \cdot A_2 + 0 \cdot A_3 = a_1 \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix}$$

Also

$$\begin{vmatrix} 0 & b_1 & 0 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = -b_1 \begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix}$$

27. Theorem. *A determinant is reduced to its principal term when each of the elements on one side of the diagonal is zero.* For, taking a determinant of the fourth order, we have successively

$$\begin{vmatrix} a_1 & 0 & 0 & 0 \\ a_2 & b_2 & 0 & 0 \\ a_3 & b_3 & c_3 & 0 \\ a_4 & b_4 & c_4 & d_4 \end{vmatrix} = a_1 \begin{vmatrix} b_2 & 0 & 0 \\ b_3 & c_3 & 0 \\ b_4 & c_4 & d_4 \end{vmatrix} = a_1 b_2 \begin{vmatrix} c_3 & 0 \\ c_4 & d_4 \end{vmatrix} = a_1 b_2 c_3 d_4.$$

This appears at once from equation (1), Art. 23, where all the terms, except the first, have zero for a factor, and therefore vanish.

28. Theorem. *To multiply a determinant by p, it suffices to multiply the elements of a row or of a column by this factor.* We have

$$p \cdot \Delta \equiv pa_1A_1 + pb_1B_1 + pc_1C_1 \equiv \begin{vmatrix} pa_1 & pb_1 & pc_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

29. If, in one of the developments

$$a_1A_1 + b_1B_1 + c_1C_1 + \cdots + l_1L_1,$$

we replace the elements, $a_1, b_1, c_1 \cdots l_1$, which appear here by those of any other row, the result is zero; the same is the case, if in one of the expressions

$$a_1A_1 + a_2A_2 + a_3A_3 + \cdots + a_nA_n,$$

we replace the elements by those of another column. For, in substituting, for example, in the place of the elements $a_1, b_1, c_1, \cdots l_1$, those of the second row $a_2, b_2, c_2, \cdots l_2$, the expression

$$a_2A_1 + b_2B_1 + c_2C_1 + \cdots + l_2L_1,$$

represents the determinant obtained by this substitution, the coefficients $A_1, B_1 \cdots L_1$ being always the minors of the first

row; and this determinant is zero, since it contains two identical rows.

That is, we have

$$a_1A_1 + b_1B_1 + c_1C'_1 + \cdots + l_1L_1 \equiv \Delta,$$

$$a_1A_1 + a_2A_2 + a_3A_3 + \cdots + a_nA_n \equiv \Delta,$$

but $$a_2A_1 + b_2B_1 + c_2C_1 + \cdots + l_2L_1 \equiv 0,$$

and other similar relations.

In general, the expression

$$a_iA_j + b_iB_j + c_iC'_j + \cdots + l_iL_j$$

represents the determinant Δ, if $j = i$; and is zero, if j is different from i.

Cor. With the notation with two subscripts, for a determinant of the nth order, this property is expressed thus: the developments

$$a_{1j}A_{1i} + a_{2j}A_{2i} + \cdots + a_{nj}A_{ni},$$

$$a_{j1}A_{i1} + a_{j2}A_{i2} + \cdots + a_{jn}A_{in},$$

represent the determinant Δ, when j is a number of the series $1, 2, 3, \cdots n$, and equal to i; while, if j is different from i, they equal zero.

30. We can always raise the order of a determinant without changing its value. Thus, after the preceding properties, we have the equalities

$$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 \\ x & a_1 & b_1 \\ y & a_2 & b_2 \end{vmatrix} = \begin{vmatrix} 1 & x & y \\ 0 & a_1 & b_1 \\ 0 & a_2 & b_2 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 & 0 \\ x & 1 & 0 & 0 \\ y & t & a_1 & b_1 \\ z & u & a_2 & b_2 \end{vmatrix}$$

and so on. The elements x, y, z, t, u being any quantities whatever.

31. Development and Evaluation of Determinants. The fundamental formula

$$\Delta \equiv a_1A_1 + b_1B_1 + c_1C_1 + \cdots + l_1L_1$$

enables us to replace a determinant of the nth order by an expression containing only determinants of the $(n-1)$th order; in this last we can substitute for A_1, B_1, C_1, $\cdots$ expressions containing only determinants of the $(n-2)$th order; in continuing in this way we finally arrive at the value of the determinant Δ. It is necessary to give some applications to indicate the steps in the different cases that may present themselves.

As we have seen, determinants of the second order are calculated directly. We have

$$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} \equiv a_1b_2 - a_2b_1.$$

$$\begin{vmatrix} 3 & 2 \\ 1 & -4 \end{vmatrix} = 3.-4 - 1.2 = -14, \quad \begin{vmatrix} 1 & 4 \\ 0 & 6 \end{vmatrix} = 6.$$

For a determinant of the third order, of which all the elements are different from zero, we would take, for example, the formula

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = a_1 \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & c_1 \\ b_3 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}$$

EXAMPLES.

1. $$\begin{vmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 4 & 5 \end{vmatrix} = 1 \begin{vmatrix} 3 & 4 \\ 4 & 5 \end{vmatrix} - 2 \begin{vmatrix} 2 & 3 \\ 4 & 5 \end{vmatrix} + 3 \begin{vmatrix} 2 & 3 \\ 3 & 4 \end{vmatrix}$$
$$= -1 - 2(-2) + 3(-1) = 0.$$

2. $$\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = x_1 \begin{vmatrix} y_2 & 1 \\ y_3 & 1 \end{vmatrix} - x_2 \begin{vmatrix} y_1 & 1 \\ y_3 & 1 \end{vmatrix} + x_3 \begin{vmatrix} y_1 & 1 \\ y_2 & 1 \end{vmatrix}$$
$$= x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2).$$

NOTE. In examples where certain elements are zero, we ought to employ the development according to the line which contains the greatest number of zero elements. Thus:

3. $\begin{vmatrix} 1 & 0 & 3 \\ 2 & 1 & 1 \\ 2 & 1 & 4 \end{vmatrix} = 1\begin{vmatrix} 1 & 1 \\ 1 & 4 \end{vmatrix} + 3\begin{vmatrix} 2 & 1 \\ 2 & 1 \end{vmatrix} = 3.$

4. $\begin{vmatrix} 1 & 2 & 0 \\ 4 & 1 & 0 \\ 1 & 3 & 2 \end{vmatrix} = 2\begin{vmatrix} 1 & 2 \\ 4 & 1 \end{vmatrix} = -14.$

5. Develop $\begin{vmatrix} a & b & e \\ b & c & f \\ e & f & g \end{vmatrix}$

6. Develop

$$\begin{vmatrix} 2 & 3 & 0 & 2 \\ 0 & 1 & 4 & 0 \\ 0 & 3 & 1 & -1 \\ 1 & 2 & 0 & 1 \end{vmatrix} = 2\begin{vmatrix} 1 & 4 & 0 \\ 3 & 1 & -1 \\ 2 & 0 & 1 \end{vmatrix} - \begin{vmatrix} 3 & 0 & 2 \\ 1 & 4 & 0 \\ 3 & 1 & -1 \end{vmatrix} = -4.$$

7. Develop $\begin{vmatrix} 1 & a & -b \\ -a & 1 & c \\ b & -c & 1 \end{vmatrix}$ *Ans.* $1 + a^2 + b^2 + c^2$.

8. Develop the determinants;

$$\begin{vmatrix} 3 & 2 & 1 \\ 5 & 6 & 7 \\ 2 & 1 & 4 \end{vmatrix}, \quad \begin{vmatrix} 0 & 2 & 5 \\ 1 & 0 & 4 \\ 3 & 6 & 0 \end{vmatrix}, \quad \begin{vmatrix} 4 & 0 & 0 \\ 2 & 6 & 8 \\ 1 & 3 & 5 \end{vmatrix}$$

9. Develop $\begin{vmatrix} 1 & 0 & 0 & 0 \\ 4 & 3 & 0 & 0 \\ 2 & 7 & 8 & 0 \\ 5 & 0 & 6 & 2 \end{vmatrix}$

10. Develop $\begin{vmatrix} 0 & c & d \\ c & 1 & \sin\alpha \\ d & \sin\alpha & 1 \end{vmatrix}$

32. Laplace's Development, — Development of a Determinant according to the Elements of Two Rows or of Two Columns. Take the determinant of the nth order:

$$\Delta \equiv \begin{vmatrix} a_1 & b_1 & c_1 & \cdots & l_1 \\ a_2 & b_2 & c_2 & \cdots & l_2 \\ a_3 & b_3 & c_3 & \cdots & l_3 \\ a_4 & b_4 & c_4 & \cdots & l_4 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ a_n & b_n & c_n & \cdots & l_n \end{vmatrix}$$

Let us consider the principal term of Δ, $a_1b_2c_3d_4\cdots l_n$; to this term there corresponds another, $-a_2b_1c_3d_4\cdots l_n$, arising from the interchange of the letters a and b. Uniting these two terms so as to put their common factor in evidence, we have

$$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} c_3d_4\cdots l_n.$$

Let a and b be fixed, and form all the possible permutations of the other letters, c, d, $\cdots l$, then the terms of Δ, which have as a factor the determinant (a_1b_2), will be represented by

$$(a_1b_2)\Sigma \pm c_3d_4e_5\cdots l_n.$$

The coefficient of (a_1b_2) is therefore the second minor of Δ obtained in suppressing the rows and columns which contain these elements. We reach an analogous conclusion for the coefficients of the determinants of the second order

$$(a_1b_3),\ (a_1b_4)\cdots(a_1b_n),\ (a_2b_3)\cdots(a_{n-1}b_n),$$

which result from the combinations two and two of the elements of the first two columns of Δ. In calling the second minors B_{12}, B_{13}, B_{14}, etc., we find the following development:

$$\Delta \equiv (a_1b_2)B_{12} + (a_1b_3)B_{13} + \cdots + (a_1b_n)B_{1n} + (a_2b_3)B_{23} + \cdots \\ + (a_{n-1}b_n)B_{(n-1)n}.$$

The number of terms of the second member is represented by

$$2\,\frac{n(n-1)}{1\cdot 2}(1\cdot 2\cdot 3 \cdots (n-2)) = n\,!$$

as it ought to be.

It is important to remark that, in the preceding formula, it is necessary to attribute to the second minors a sign in conformity with the value of Δ.

In the first place, the second minor B_{12} ought to have the positive sign. Finally to fix the sign of the others, it is necessary by the interchange of lines, to lead the coefficients of the determinant of the second order to the first two places, in preserving always the order of the indices. Thus, the second minor B_{13} would be negative, because one interchange of two lines is necessary to lead a_3, b_3 to the place of a_2, b_2; the minor B_{23} would be positive, for there is necessary one interchange to lead a_2, b_2 to the first row, and another to lead a_3, b_3 to the second row. And so on for the others.

Using these principles, let us develop the determinant of the fifth order:

$$\Delta \equiv \begin{vmatrix} a_1 & b_1 & c_1 & d_1 & e_1 \\ a_2 & b_2 & c_2 & d_2 & e_2 \\ a_3 & b_3 & c_3 & d_3 & e_3 \\ a_4 & b_4 & c_4 & d_4 & e_4 \\ a_5 & b_5 & c_5 & d_5 & e_5 \end{vmatrix}$$

according to the elements of the first two columns. It will become, with the abridged notation (Art. 17),

$$\Delta \equiv (a_1b_2)(c_3d_4e_5) - (a_1b_3)(c_2d_4e_5) + (a_1b_4)(c_2d_3e_5) - (a_1b_5)(c_2d_3e_4) \\ + (a_2b_3)(c_1d_4e_5) - (a_2b_4)(c_1d_3e_5) + (a_2b_5)(c_1d_3e_4) + (a_3b_4)(c_1d_2e_5) \\ - (a_3b_5)(c_1d_2e_4) + (a_4b_5)(c_1d_2e_3).$$

The same mode of development exists relatively to any two rows or any two columns.

This development may be applied to the calculation of a determinant of the fourth order, the expansion giving only determinants of the second order. We have

$$\begin{vmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \\ a_4 & b_4 & c_4 & d_4 \end{vmatrix} = \begin{array}{l} (a_1b_2)(c_3d_4) - (a_1c_2)(b_3d_4) + (a_1d_2)(b_3c_4) \\ + (b_1c_2)(a_3d_4) - (b_1d_2)(a_3c_4) + (c_1d_2)(a_3b_4). \end{array}$$

1. For example, calculate

$$\Delta \equiv \begin{vmatrix} 3 & 1 & 1 & 2 \\ 1 & 5 & 0 & 3 \\ 2 & -2 & 1 & 6 \\ 0 & 4 & -5 & 3 \end{vmatrix}$$

Developing, as above, we have

$$\Delta \equiv \begin{vmatrix} 3 & 1 \\ 1 & 5 \end{vmatrix} \cdot \begin{vmatrix} 1 & 6 \\ -5 & 3 \end{vmatrix} - \begin{vmatrix} 3 & 1 \\ 1 & 0 \end{vmatrix} \cdot \begin{vmatrix} -2 & 6 \\ 4 & 3 \end{vmatrix} + \begin{vmatrix} 3 & 2 \\ 1 & 3 \end{vmatrix} \cdot \begin{vmatrix} -2 & 1 \\ 4 & -5 \end{vmatrix}$$
$$+ \begin{vmatrix} 1 & 1 \\ 5 & 0 \end{vmatrix} \cdot \begin{vmatrix} 2 & 6 \\ 0 & 3 \end{vmatrix} - \begin{vmatrix} 1 & 2 \\ 5 & 3 \end{vmatrix} \cdot \begin{vmatrix} 2 & 1 \\ 0 & -5 \end{vmatrix} + \begin{vmatrix} 1 & 2 \\ 0 & 3 \end{vmatrix} \cdot \begin{vmatrix} 2 & -2 \\ 0 & 4 \end{vmatrix}$$

or

$$\Delta = 14 \cdot 33 + 1 \cdot -30 + 7 \cdot 6 - 5 \cdot 6 + 7 \cdot -10 + 3 \cdot 8 = 398.$$

2. Calculate

$$\Delta \equiv \begin{vmatrix} 1 & 2 & -1 & 0 \\ 2 & 1 & 2 & 0 \\ 0 & 3 & 5 & 2 \\ 1 & 0 & 1 & 2 \end{vmatrix} = \begin{vmatrix} 1 & 2 \\ 2 & 1 \end{vmatrix} \cdot \begin{vmatrix} 5 & 2 \\ 1 & 2 \end{vmatrix} - \begin{vmatrix} 1 & -1 \\ 2 & 2 \end{vmatrix} \cdot \begin{vmatrix} 3 & 2 \\ 0 & 2 \end{vmatrix} + \begin{vmatrix} 2 & -1 \\ 1 & 2 \end{vmatrix} \cdot \begin{vmatrix} 0 & 2 \\ 1 & 2 \end{vmatrix} = -58.$$

3.
$$\begin{vmatrix} a_1 & b_1 & 0 & 0 \\ a_2 & b_2 & 0 & 0 \\ a_3 & b_3 & c_3 & d_3 \\ a_4 & b_4 & c_4 & d_4 \end{vmatrix} = (a_1b_2)(c_3d_4).$$

The following example illustrates how the operation may be shortened by first bringing the zero elements into consecutive positions.

4. $$\begin{vmatrix} a_1 & b_1 & c_1 & d_1 \\ 0 & b_2 & c_2 & 0 \\ a_3 & b_3 & c_3 & d_3 \\ 0 & b_4 & c_4 & 0 \end{vmatrix} = \begin{vmatrix} a_1 & d_1 & b_1 & c_1 \\ 0 & 0 & b_2 & c_2 \\ a_3 & d_3 & b_3 & c_3 \\ 0 & 0 & b_4 & c_4 \end{vmatrix} = - \begin{vmatrix} 0 & 0 & b_2 & c_2 \\ 0 & 0 & b_4 & c_4 \\ a_1 & d_1 & b_1 & c_1 \\ a_3 & d_3 & b_3 & c_3 \end{vmatrix} = -(b_2c_4)(a_1d_3).$$

5. Prove the identity

$$\begin{vmatrix} a_1 & b_1 & c_1 & x_1 & y_1 & z_1 \\ a_2 & b_2 & c_2 & x_2 & y_2 & z_2 \\ a_3 & b_3 & c_3 & x_3 & y_3 & z_3 \\ 0 & 0 & 0 & \alpha_1 & \beta_1 & \gamma_1 \\ 0 & 0 & 0 & \alpha_2 & \beta_2 & \gamma_2 \\ 0 & 0 & 0 & \alpha_3 & \beta_3 & \gamma_3 \end{vmatrix} \equiv \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \begin{vmatrix} \alpha_1 & \beta_1 & \gamma_1 \\ \alpha_2 & \beta_2 & \gamma_2 \\ \alpha_3 & \beta_3 & \gamma_3 \end{vmatrix}$$

This appears by expanding the determinant in terms of the minors formed from the first three columns, for it is evident that all these minors vanish (having at least one row of ciphers) except one, viz. $(a_1 b_2 c_3)$.

In general it appears in the same way, that if a determinant of the $2m$th order contains in any position a square of m^2 ciphers, it can be expressed as the product of two determinants of the mth order.

This is known as Laplace's Method,* and can readily be extended to the general case. Let any number p of columns be taken, and all possible minors formed by taking p rows of these columns. Each of these minors is then to be multiplied by the complementary minor, and the determinant expressed as the sum of all such products with their proper signs.

* Pierre Simon Laplace (1749-1827), the great French mathematician and astronomer

ADDITION OF DETERMINANTS.

33. Theorem. *If every element in any row (or column) can be resolved into the sum of two others, the determinant can be resolved into the sum of two others.*

Suppose the elements of the first column to be $a_1+\alpha_1$, $a_2+\alpha_2$, $a_3+\alpha_3$ etc. Substituting these in the expansion of Art. 23, equation 2, we have

$$\Delta \equiv (a_1+\alpha_1)A_1+(a_2+\alpha_2)A_2+(a_3+\alpha_3)A_3+\text{etc.}$$

$$\equiv a_1A_1+a_2A_2+a_3A_3+\cdots\text{etc.}+\alpha_1A_1+\alpha_2A_2+\alpha_3A_3+\text{etc.};$$

or,

$$\begin{vmatrix} a_1+\alpha_1 & b_1 & c_1 & \cdots \\ a_2+\alpha_2 & b_2 & c_2 & \cdots \\ a_3+\alpha_3 & b_3 & c_3 & \cdots \\ \cdot & \cdot & \cdot & \cdot \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & c_1 & \cdots \\ a_2 & b_2 & c_2 & \cdots \\ a_3 & b_3 & c_3 & \cdots \\ \cdot & \cdot & \cdot & \cdot \end{vmatrix} + \begin{vmatrix} \alpha_1 & b_1 & c_1 & \cdots \\ \alpha_2 & b_2 & c_2 & \cdots \\ \alpha_3 & b_3 & c_3 & \cdots \\ \cdot & \cdot & \cdot & \cdot \end{vmatrix}$$

which proves the proposition.

Similarly, the determinant

$$\begin{vmatrix} a_1+\alpha_1 & b_1+\beta_1 & c_1 \\ a_2+\alpha_2 & b_2+\beta_2 & c_2 \\ a_3+\alpha_3 & b_3+\beta_3 & c_3 \end{vmatrix}$$

is equal to the sum of the four determinants

$$(a_1b_2c_3)+(\alpha_1b_2c_3)+(a_1\beta_2c_3)+(\alpha_1\beta_2c_3).$$

In like manner it follows that if each of the elements of one column consists of the algebraical sum of any number of terms, the determinant can be resolved into the sum of a corresponding number of determinants. For example:

$$\begin{vmatrix} a_1-\alpha_1+\alpha_1' & b_1 & c_1 \\ a_2-\alpha_2+\alpha_2' & b_2 & c_2 \\ a_3-\alpha_3+\alpha_3' & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} - \begin{vmatrix} \alpha_1 & b_1 & c_1 \\ \alpha_2 & b_2 & c_2 \\ \alpha_3 & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} \alpha_1' & b_1 & c_1 \\ \alpha_2' & b_2 & c_2 \\ \alpha_3' & b_3 & c_3 \end{vmatrix}$$

And, in general, if one column (or row) consists of the algebraic sum of m others, a second column (or row) of the sum of n others, a third of the sum of p others, etc., the determinant can be resolved into the sum of $mnp\cdots$, etc., others.

34. Theorem. *If the elements of one row (or column) are equal to the sums of the corresponding elements of the other rows (or columns) multiplied by constant factors, the determinant vanishes.*

For it can then be resolved into the sum of a number of determinants which separately vanish. For example,

$$\begin{vmatrix} ma_1+nb_1 & a_1 & b_1 \\ ma_2+nb_2 & a_2 & b_2 \\ ma_3+nb_3 & a_3 & b_3 \end{vmatrix} = m\begin{vmatrix} a_1 & a_1 & b_1 \\ a_2 & a_2 & b_2 \\ a_3 & a_3 & b_3 \end{vmatrix} + n\begin{vmatrix} b_1 & a_1 & b_1 \\ b_2 & a_2 & b_2 \\ b_3 & a_3 & b_3 \end{vmatrix},$$

and each of the latter determinants vanishes (Art. 20).

35. Theorem. *A determinant is unchanged when to each element of any row or column are added those of several other rows or columns, multiplied respectively by constant factors.*

For when the determinant is resolved into the sum of others, as in Art. 33, the determinants in which the added lines occur all vanish, since each of them must, when the constant factor is removed, contain two identical lines.

Thus, for example,

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \equiv \begin{vmatrix} a_1+mb_1+nc_1 & b_1 & c_1 \\ a_2+mb_2+nc_2 & b_2 & c_2 \\ a_3+mb_3+nc_3 & b_3 & c_3 \end{vmatrix}.$$

This is evident since, when the second determinant is expressed as the sum of three others, the two arising from the added columns vanish identically (Art. 34).

This proposition will be found very useful in the evaluation of determinants.

EXAMPLES.

1. Find the value of the determinant

$$\begin{vmatrix} 1 & 2 & 4 \\ 2 & 3 & 7 \\ 3 & 4 & 10 \end{vmatrix}$$

Subtracting the elements of the first column from those of the second, and three times the elements of the first column from those of the third, we obtain

$$\begin{vmatrix} 1 & 1 & 1 \\ 2 & 1 & 1 \\ 3 & 1 & 1 \end{vmatrix}$$

which is identically equal to zero.

2. Evaluate
$$\begin{vmatrix} -1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 1 \\ 1 & 1 & -1 & 1 \\ 1 & 1 & 1 & -1 \end{vmatrix}$$

3. Evaluate

$$\begin{vmatrix} 7 & -2 & 0 & 5 \\ -2 & 6 & -2 & 2 \\ 0 & -2 & 5 & 3 \\ 5 & 2 & 3 & 4 \end{vmatrix} = \begin{vmatrix} 7 & -2 & 0 & 5 \\ 19 & 0 & -2 & 17 \\ -7 & 0 & 5 & -2 \\ 12 & 0 & 3 & 9 \end{vmatrix} = 2\begin{vmatrix} 19 & -2 & 17 \\ -7 & 5 & -2 \\ 12 & 3 & 9 \end{vmatrix} = -972.$$

Here the first transformation is obtained by adding to the second row three times the first, subtracting the first from the third row, and adding the first to the fourth row.

4. Calculate the determinant

$$\Delta \equiv \begin{vmatrix} 1 & 15 & 14 & 4 \\ 12 & 6 & 7 & 9 \\ 8 & 10 & 11 & 5 \\ 13 & 3 & 2 & 16 \end{vmatrix}$$

The first sixteen natural numbers are arranged here in what is called a "magic square," *i.e.* the sum of all the figures in any row or any column is constant. In general, for a square of the first n^2 numbers, this sum is $\frac{1}{2}n(n^2+1)$. Determinants of this kind can be at once reduced one degree.

Here adding the last three columns to the first, and subtracting the last row from each of the others, we have

$$\Delta \equiv 34\begin{vmatrix} 1 & 15 & 14 & 4 \\ 1 & 6 & 7 & 9 \\ 1 & 10 & 11 & 5 \\ 1 & 3 & 2 & 16 \end{vmatrix} = 34\begin{vmatrix} 0 & 12 & 12 & -12 \\ 0 & 3 & 5 & -7 \\ 0 & 7 & 9 & -11 \\ 1 & 3 & 2 & 16 \end{vmatrix} = -34\times 12\begin{vmatrix} 1 & 1 & -1 \\ 3 & 5 & -7 \\ 7 & 9 & -11 \end{vmatrix}$$

and subtracting the second row from the last row, it is evident that the reduced determinant vanishes; hence $\Delta = 0$.

5. Calculate the determinant formed by the first nine natural numbers arranged in a magic square:

$$\begin{vmatrix} 4 & 9 & 2 \\ 3 & 5 & 7 \\ 8 & 1 & 6 \end{vmatrix}$$ *Ans.* 360.

6. Calculate

$$\begin{vmatrix} 2 & 3 & 8 \\ 4 & 6 & 4 \\ 6 & 12 & 4 \end{vmatrix}$$ *Ans.* 72.

7. $$\Delta \equiv \begin{vmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & z^2 & y^2 \\ 1 & z^2 & 0 & x^2 \\ 1 & y^2 & x^2 & 0 \end{vmatrix} = \begin{vmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & z^2 & y^2 \\ 1 & z^2 & -z^2 & x^2-z^2 \\ 1 & y^2 & x^2-y^2 & -y^2 \end{vmatrix} = -\begin{vmatrix} 1 & z^2 & y^2 \\ 1 & -z^2 & x^2-z^2 \\ 1 & x^2-y^2 & -y^2 \end{vmatrix}$$

Here, to obtain the second determinant, we subtract the second column from each of the following ones. In the re-

duced determinant, subtracting the first row from each of the following, we find

$$\Delta \equiv -\begin{vmatrix} 1 & z^2 & y^2 \\ 0 & -2z^2 & x^2-z^2-y^2 \\ 0 & x^2-y^2-z^2 & -2y^2 \end{vmatrix} = -\begin{vmatrix} 2z^2 & y^2+z^2-x^2 \\ y^2+z^2-x^2 & 2y^2 \end{vmatrix}$$

$$= (y^2+z^2-x^2)^2 - 4y^2z^2$$

$$= (y^2+z^2-x^2+2yz)(y^2+z^2-x^2-2yz)$$

$$= -(x+y+z)(y+z-x)(z+x-y)(x+y-z).$$

8. Evaluate the determinant

$$\begin{vmatrix} 1 & 1 & 1 & 4 \\ 2 & 4 & 1 & 8 \\ 4 & 1 & 2 & 13 \\ 2 & 4 & 2 & 11 \end{vmatrix}$$

Ans. -15.

9. Evaluate
$$\begin{vmatrix} 2 & 2 & 2 & 10 \\ 1 & -1 & -1 & 5 \\ 3 & -3 & 3 & -15 \\ 1 & 1 & -1 & -5 \end{vmatrix}$$

10. Evaluate

$$\begin{vmatrix} a & b & c & d \\ -a & b & a & \beta \\ -a & -b & c & \gamma \\ -a & -b & -c & d \end{vmatrix} = \begin{vmatrix} a & b & c & d \\ 0 & 2b & c+a & d+\beta \\ 0 & 0 & 2c & d+\gamma \\ 0 & 0 & 0 & 2d \end{vmatrix} = 2^3abcd.$$

11. Evaluate
$$\Delta \equiv \begin{vmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & a & b \\ 1 & a & 0 & c \\ 1 & b & c & 0 \end{vmatrix}$$

Ans. $\Delta \equiv a^2+b^2+c^2-2bc-2ac-2ab.$

MULTIPLICATION OF DETERMINANTS.

36. Theorem. *The product of two determinants of any order is itself a determinant of the same order.*

We shall prove this for two determinants of the third order, and, from the nature of the proof, it will be evident that it is equally applicable in general.

We propose to show that the product of the two determinants

$$A \equiv (a_1b_2c_3) \text{ and } B \equiv (\alpha_1\beta_2\gamma_3) \text{ is } P \equiv$$

$$\begin{vmatrix} a_1\alpha_1+b_1\beta_1+c_1\gamma_1 & a_1\alpha_2+b_1\beta_2+c_1\gamma_2 & a_1\alpha_3+b_1\beta_3+c_1\gamma_3 \\ a_2\alpha_1+b_2\beta_1+c_2\gamma_1 & a_2\alpha_2+b_2\beta_2+c_2\gamma_2 & a_2\alpha_3+b_2\beta_3+c_2\gamma_3 \\ a_3\alpha_1+b_3\beta_1+c_3\gamma_1 & a_3\alpha_2+b_3\beta_2+c_3\gamma_2 & a_3\alpha_3+b_3\beta_3+c_3\gamma_3 \end{vmatrix} \quad (1)$$

whose elements are the sums of the products of the elements in any row of $(a_1b_2c_3)$ by the corresponding elements in any row of $(\alpha_1\beta_2\gamma_3)$. The determinant P can evidently (Art. 33) be expanded into the sum of twenty-seven others.

The following proof of this theorem is derived from Laplace's method of development already explained (Art. 32).

The product of the two determinants, A, B, is (see Ex. 5, Art. 32) plainly equal to the determinant

$$\begin{vmatrix} a_1 & b_1 & c_1 & 0 & 0 & 0 \\ a_2 & b_2 & c_2 & 0 & 0 & 0 \\ a_3 & b_3 & c_3 & 0 & 0 & 0 \\ -1 & 0 & 0 & \alpha_1 & \alpha_2 & \alpha_3 \\ 0 & -1 & 0 & \beta_1 & \beta_2 & \beta_3 \\ 0 & 0 & -1 & \gamma_1 & \gamma_2 & \gamma_3 \end{vmatrix} \quad . \quad . \quad . \quad . \quad (2)$$

In this determinant add to the fourth column the sum of the first multiplied by α_1, the second by β_1, and the third by γ_1; add to the fifth column the sum of the first multiplied by α_2, the second by β_2, and the third by γ_2; and add to the sixth

column the sum of the first multiplied by α_3, the second by β_3, and the third by γ_3. The determinant (2) becomes then

$$\begin{vmatrix} a_1 & b_1 & c_1 & a_1\alpha_1+b_1\beta_1+c_1\gamma_1 & a_1\alpha_2+b_1\beta_2+c_1\gamma_2 & a_1\alpha_3+b_1\beta_3+c_1\gamma_3 \\ a_2 & b_2 & c_2 & a_2\alpha_1+b_2\beta_1+c_2\gamma_1 & a_2\alpha_2+b_2\beta_2+c_2\gamma_2 & a_2\alpha_3+b_2\beta_3+c_2\gamma_3 \\ a_3 & b_3 & c_3 & a_3\alpha_1+b_3\beta_1+c_3\gamma_1 & a_3\alpha_2+b_3\beta_2+c_3\gamma_2 & a_3\alpha_3+b_3\beta_3+c_3\gamma_3 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \end{vmatrix}$$

And this is, by Ex. 5, Art. 32, equal to the product, with the proper sign (which in this case is evidently $-$), of the determinant

$$\begin{vmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{vmatrix} \text{ (which is equal to } -1)$$

by the complementary minor, which is the P of this article. Hence, the theorem

$$A \times B = P.$$

Cor. Two determinants of different orders may be multiplied together by raising the lower determinant to the order of the higher (Art. 30), and then applying the above rule. Thus:

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \times \begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \times \begin{vmatrix} 1 & 0 & 0 \\ 0 & x_1 & y_1 \\ 0 & x_2 & y_2 \end{vmatrix}$$

$$= \begin{vmatrix} a_1 & b_1x_1+c_1y_1 & b_1x_2+c_1y_2 \\ a_2 & b_2x_1+c_2y_1 & b_2x_2+c_2y_2 \\ a_3 & b_3x_1+c_3y_1 & b_3x_2+c_3y_2 \end{vmatrix}$$

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37. Euler's Theorem. *The product of two numbers, each the sum of four squares, is itself the sum of four squares.*

By Laplace's method of development, we readily prove the following identity:

$$\begin{vmatrix} a & b & c & d \\ -b & a & -d & c \\ -c & d & a & -b \\ -d & -c & b & a \end{vmatrix} = (a^2+b^2+c^2+d^2)^2 \quad . \quad (1)$$

Similarly,

$$\begin{vmatrix} \alpha & \beta & \gamma & \delta \\ -\beta & \alpha & -\delta & \gamma \\ -\gamma & \delta & \alpha & -\beta \\ -\delta & -\gamma & \beta & \alpha \end{vmatrix} = (\alpha^2+\beta^2+\gamma^2+\delta^2)^2 \quad . \quad (2)$$

Now multiply equations (1) and (2) together, member for member.

Letting

$$a\alpha + b\beta + c\gamma + d\delta = A,$$
$$-a\beta + b\alpha - c\delta + d\gamma = B,$$
$$-a\gamma + b\delta + c\alpha - d\beta = C,$$
$$-a\delta - b\gamma + c\beta + d\alpha = D,$$

the product of the left-hand members may be written:

$$\begin{vmatrix} A & B & C & D \\ -B & A & -D & C \\ -C & D & A & -B \\ -D & -C & B & A \end{vmatrix} = (A^2+B^2+C^2+D^2)^2 \quad . \quad (3)$$

Therefore

$$(a^2+b^2+c^2+d^2)(\alpha^2+\beta^2+\gamma^2+\delta^2) = (A^2+B^2+C^2+D^2),$$

which is the theorem.*

* This theorem is due to, and named after, the Swiss mathematician Leonhard Euler (1707–1783).

EXAMPLES.

1. Find the product of the two determinants

$$\begin{vmatrix} 1 & a_1 & b_1 \\ 1 & a_2 & b_2 \\ 1 & a_3 & b_3 \end{vmatrix}, \quad \begin{vmatrix} 1 & 1 & 1 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{vmatrix}$$

2. Find the value of $\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}^2$

3. Find the product of the two determinants

$$\begin{vmatrix} 1 & 3 & 1 & 0 \\ 3 & 2 & 3 & 1 \\ 0 & 2 & 5 & 1 \\ 2 & 2 & 2 & 3 \end{vmatrix}, \quad \begin{vmatrix} 0 & 1 & 0 & 2 \\ 2 & 0 & 1 & 0 \\ 1 & 0 & 2 & 0 \\ 0 & 2 & 0 & 1 \end{vmatrix}$$

38. Rectangular Arrays. Arrays in which the number of rows is not equal to the number of columns are called *rectangular*. The common notation for rectangular arrays, or *matrices*, as they are called, is:

$$\begin{Vmatrix} a_1 & b_1 \\ a_2 & b_2 \\ a_3 & b_3 \end{Vmatrix}, \quad \begin{Vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{Vmatrix}$$

Rectangular arrays do not themselves represent any definite function; but if two such arrays of the same dimensions are given, we can derive from them by the multiplication theorem of Art. 36 a determinant whose value we proceed to investigate.

(1) *When the number of columns exceeds the number of rows.*

THEOREM.* *The "product" of two rectangular arrays of the same dimensions is equal to the sum of the products of all possi-*

* By the so-called "product" here and the multiplication of two rectangular arrays in the following theorem, we simply mean that the *process* of Art. 36 is employed; of course, as matrices are not functions, they cannot really be multiplied together.

ble determinants which can be formed from our array (by taking a number of columns equal to the number of rows) multiplied by the corresponding determinants formed from the other array.

To prove this, take any two rectangular arrays,

$$\begin{Vmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \end{Vmatrix} \quad (1), \qquad \begin{Vmatrix} \alpha_1 & \beta_1 & \gamma_1 & \delta_1 \\ \alpha_2 & \beta_2 & \gamma_2 & \delta_2 \end{Vmatrix} \quad (2)$$

and perform on these a process similar to that employed in multiplying two determinants. We thus obtain the determinant

$$\begin{vmatrix} a_1\alpha_1 + b_1\beta_1 + c_1\gamma_1 + d_1\delta_1 & a_1\alpha_2 + b_1\beta_2 + c_1\gamma_2 + d_1\delta_2 \\ a_2\alpha_1 + b_2\beta_1 + c_2\gamma_1 + d_2\delta_1 & a_2\alpha_2 + b_2\beta_2 + c_2\gamma_2 + d_2\delta_2 \end{vmatrix}$$

The value of this is easily found to be

$$(a_1b_2)(\alpha_1\beta_2) + (a_1c_2)(\alpha_1\gamma_2) + (a_1d_2)(\alpha_1\delta_2) + (b_1c_2)(\beta_1\gamma_2)$$
$$+ (b_1d_2)(\beta_1\delta_2) + (c_1d_2)(\gamma_1\delta_2).$$

Hence the theorem. This proof can be easily generalized.

(2) *When the number of rows exceeds the number of columns.*

THEOREM. *In this case, the determinant resulting from the multiplication* (so called) *of the two arrays vanishes.*

Take, for example, the two arrays,

$$\begin{Vmatrix} a_1 & b_1 \\ a_2 & b_2 \\ a_3 & b_3 \end{Vmatrix} \quad (1), \qquad \begin{Vmatrix} \alpha_1 & \beta_1 \\ \alpha_2 & \beta_2 \\ \alpha_3 & \beta_3 \end{Vmatrix} \quad (2).$$

Performing the process of multiplication, we have the determinant

$$\begin{vmatrix} a_1\alpha_1 + b_1\beta_1 & a_1\alpha_2 + b_1\beta_2 & a_1\alpha_3 + b_1\beta_3 \\ a_2\alpha_1 + b_2\beta_1 & a_2\alpha_2 + b_2\beta_2 & a_2\alpha_3 + b_2\beta_3 \\ a_3\alpha_1 + b_3\beta_1 & a_3\alpha_2 + b_3\beta_2 & a_3\alpha_3 + b_3\beta_3 \end{vmatrix}$$

This determinant is obviously the same as would arise if a column of ciphers were added to each of the given arrays, and the determinants so formed then multiplied. It follows that the *determinant* vanishes.

In an exactly similar way, we can prove the general theorem.

EXAMPLES.

1. From the two arrays

$$\begin{vmatrix} 1 & 1 & 1 \\ \alpha & \beta & \gamma \end{vmatrix} (1), \qquad \begin{vmatrix} 1 & 1 & 1 \\ \alpha & \beta & \gamma \end{vmatrix} (2),$$

prove

$$\begin{vmatrix} 3 & \alpha+\beta+\gamma \\ \alpha+\beta+\gamma & \alpha^2+\beta^2+\gamma^2 \end{vmatrix} \equiv (\alpha-\beta)^2+(\alpha-\gamma)^2+(\beta-\gamma)^2.$$

2. By squaring the array

$$\left\| \begin{matrix} a & b & c \\ d & e & f \end{matrix} \right\|$$

prove

$$(a^2+b^2+c^2)(d^2+e^2+f^2) \equiv (ad+be+cf)^2+(ae-bd)^2+(cd-af)^2 + (bf-ce)^2.$$

39. Reciprocal Determinants. The *first minors* (with their proper signs) $A_1, B_1, C_1, \cdots A_2, B_2$ etc. (Art. 22), which occur in the expansion of a determinant are called *inverse elements;* and the determinant formed with them as elements is called the inverse or *reciprocal* of the original determinant. The following theorem gives a useful relation connecting the two determinants:

THEOREM. *The reciprocal of any determinant of the* nth *order is equal to the* $(n-1)$th *power of the given determinant.*

Let the reciprocal of Δ be denoted by Δ', and multiply the two determinants

$$\Delta \equiv \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}, \qquad \Delta' \equiv \begin{vmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \\ A_3 & B_3 & C_3 \end{vmatrix}$$

All the elements of the resulting determinant except those in the diagonal vanish (Art. 29); and the result is

$$\Delta\Delta' = \begin{vmatrix} \Delta & 0 & 0 \\ 0 & \Delta & 0 \\ 0 & 0 & \Delta \end{vmatrix} = \Delta^3;$$

whence $\qquad \Delta' = \Delta^2.$

From the nature of the above proof, it is evident that the process here employed in a particular case is equally applicable in general; giving for a determinant of the nth order

$$\Delta\Delta' = \Delta^n, \quad \text{or} \quad \Delta' = \Delta^{n-1}.$$

EXAMPLES.

1. If Δ' = the reciprocal of the determinant

$$\Delta \equiv \begin{vmatrix} 1 & 2 & 3 \\ 3 & 1 & 4 \\ 6 & 4 & 5 \end{vmatrix}, \text{ show that } \Delta' \equiv \begin{vmatrix} -11 & 9 & 6 \\ 2 & -13 & 8 \\ 5 & 5 & -5 \end{vmatrix}$$

and, hence, verify the formula $\Delta' = \Delta^2$.

2. Form the reciprocal of the determinant

$$\Delta \equiv \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}$$

CHAPTER III.

APPLICATIONS AND SPECIAL FORMS OF DETERMINANTS.

APPLICATIONS OF DETERMINANTS.

In Arts. 9, 11, 12, and 14, we have seen how the work of solving simple linear equations of two or three variables may be abbreviated by the use of the determinant notation. We shall now extend these principles, and proceed to investigate some of the fundamental properties of systems of equations.

40. First, taking a special case, let it be required to solve the simultaneous linear equations,

$$\begin{aligned} a_1'x' + a_1''x'' + a_1'''x''' &= u_1 \\ a_2'x' + a_2''x'' + a_2'''x''' &= u_2 \\ a_3'x' + a_3''x'' + a_3'''x''' &= u_3 \end{aligned} \quad . \quad . \quad . \quad . \quad . \quad (1)$$

$$\Delta = \begin{vmatrix} a_1' & a_1'' & a_1''' \\ a_2' & a_2'' & a_2''' \\ a_3' & a_3'' & a_3''' \end{vmatrix} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

is called the determinant of this system of equations.

By Art. 29, we have:

$$\begin{aligned} A_1'a_1' + A_2'a_2' + A_3'a_3' &= \Delta, \\ A_1'a_1'' + A_2'a_2'' + A_3'a_3'' &= 0, \\ A_1'a_1''' + A_2'a_2''' + A_3'a_3''' &= 0, \end{aligned} \quad . \quad . \quad . \quad . \quad . \quad (3)$$

If now we add the equations (1) after having multiplied them respectively by A_1', A_2', A_3', the coefficient of x' would

become Δ, and those of x'' and x''' become zero. Hence we have

$$\Delta x' = A_1'u_1 + A_2'u_2 + A_3'u_3$$

$$= \begin{vmatrix} u_1 & a_1'' & a_1''' \\ u_2 & a_2'' & a_2''' \\ u_3 & a_3'' & a_3''' \end{vmatrix}$$

or,
$$x' = \frac{\begin{vmatrix} u_1 & a_1'' & a_1''' \\ u_2 & a_2'' & a_2''' \\ u_3 & a_3'' & a_3''' \end{vmatrix}}{\begin{vmatrix} a_1' & a_1'' & a_1''' \\ a_2' & a_2'' & a_2''' \\ a_3' & a_3'' & a_3''' \end{vmatrix}}$$

The values of x'' and x''' may be found in the same manner.

We proceed in exactly the same way to solve the general case, as follows.

41. Let the given system of simultaneous linear equations be

$$\begin{aligned} a_1'x' + a_1''x'' + \cdots + a_1^{(t)}x^{(t)} + \cdots + a_1^{(n)}x^{(n)} &= u_1 \\ a_2'x' + a_2''x'' + \cdots + a_2^{(t)}x^{(t)} + \cdots + a_2^{(n)}x^{(n)} &= u_2 \quad . \quad . \quad (1) \\ . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . & \\ a_n'x' + a_n''x'' + \cdots + a_n^{(t)}x^{(t)} + \cdots + a_n^{(n)}x^{(n)} &= u_n \end{aligned}$$

where the number of unknown quantities is the same as the number of equations. Let us form the determinant of this system of equations

$$\Delta \equiv \begin{vmatrix} a_1' & a_1'' & \cdots & a_1^{(t)} & \cdots & a_1^{(n)} \\ a_2' & a_2'' & \cdots & a_2^{(t)} & \cdots & a_2^{(n)} \\ . & . & . & . & . & . \\ a_n' & a_n'' & \cdots & a_n^{(t)} & \cdots & a_n^{(n)} \end{vmatrix} \quad . \quad . \quad . \quad . \quad . \quad (2)$$

and let $A_s^{(t)}$ be the coefficient of $a_s^{(t)}$ in this determinant.

The sum

$$A_1^{(i)}a_1^{(j)} + A_2^{(i)}a_2^{(j)} + \cdots + A_i^{(i)}a_i^{(j)} + \cdots + A_n^{(i)}a_n^{(j)} \quad . \quad . \quad . \quad (3)$$

is equal to Δ for $j = i$, and is zero for all values of j different from i. (Compare Art. 29, Cor.)

If now we add the equations (1), after having multiplied them respectively by

$$A_1^{(i)},\ A_2^{(i)},\ \cdots A_n^{(i)},$$

the coefficient of $x^{(i)}$ is equal to Δ, and those of all the other unknown quantities vanish. We have therefore

$$\Delta x^{(i)} = A_1^{(i)}u_1 + A_2^{(i)}u_2 + \cdots + A_n^{(i)}u_n$$

$$= \begin{vmatrix} a_1' \cdots a_1^{(i-1)} & u_1 & a_1^{(i+1)} \cdots a_1^{(n)} \\ a_2' \cdots a_2^{(i-1)} & u_2 & a_2^{(i+1)} \cdots a_2^{(n)} \\ \cdot\ \cdot\ \cdot\ \cdot & \cdot & \cdot\ \cdot\ \cdot\ \cdot \\ a_n' \cdots a_n^{(i-1)} & u_n & a_n^{(i+1)} \cdots a_n^{(n)} \end{vmatrix} \quad . \quad . \quad . \quad (4)$$

the second member being what Δ becomes, when we replace the coefficients

$$a_1^{(i)},\ a_2^{(i)},\ \cdots,\ a_n^{(i)}$$

of $x^{(i)}$ by the second members of the corresponding given equations

$$u_1,\ u_2,\ \cdots,\ u_n.$$

As long as Δ is different from zero, this formula gives for the n unknown quantities finite and determinate values.

Cor. If, for brevity, we denote the numerator of the fraction giving the value of $x^{(i)}$ (Equation 4) by $\delta^{(i)}$, then, with this notation, we would have:

$$x' = \frac{\delta'}{\Delta},\quad x'' = \frac{\delta''}{\Delta},\ \cdots\ x^{(n)} = \frac{\delta^{(n)}}{\Delta}.$$

Now, if $\Delta = 0$, and $\delta', \delta'', \cdots \delta^{(n)}$ are not zero, then the values of the unknowns are infinite.

If $\Delta = 0$, and at the same time $\delta' = 0$, $\delta'' = 0$, etc., then the values of the unknowns are indeterminate. This would be the case if $u_1 = 0$, $u_2 = 0$, $\cdots u_n = 0$.

EXAMPLES.

1. Solve the equations

$$\begin{aligned}
x + y + z + t + u &= 5\\
x + y + z + t + v &= 3\\
x + y + z + u + v &= 1\\
x + y + t + u + v &= 7\\
x + z + t + u + v &= 9\\
y + z + t + u + v &= 11
\end{aligned}$$

Here there are six equations and six unknowns, and as Δ is not zero, as we find by calculation, there is a solution. We first calculate Δ, and then the determinants which we may call $\delta_x, \delta_y, \delta_z, \delta_t, \delta_u, \delta_v$.

$$\Delta = \begin{vmatrix} 1 & 1 & 1 & 1 & 1 & 0\\ 1 & 1 & 1 & 1 & 0 & 1\\ 1 & 1 & 1 & 0 & 1 & 1\\ 1 & 1 & 0 & 1 & 1 & 1\\ 1 & 0 & 1 & 1 & 1 & 1\\ 0 & 1 & 1 & 1 & 1 & 1 \end{vmatrix} = \begin{vmatrix} 0 & 0 & 0 & 0 & 1 & -1\\ 0 & 0 & 0 & 1 & -1 & 0\\ 0 & 0 & 1 & -1 & 0 & 0\\ 0 & 1 & -1 & 0 & 0 & 0\\ 1 & -1 & 0 & 0 & 0 & 0\\ 0 & 1 & 1 & 1 & 1 & 1 \end{vmatrix}$$

$$= \begin{vmatrix} 0 & 0 & 0 & 0 & 0 & -1\\ 0 & 0 & 0 & 0 & -1 & 0\\ 0 & 0 & 0 & -1 & 0 & 0\\ 0 & 0 & -1 & 0 & 0 & 0\\ 0 & -1 & 0 & 0 & 0 & 0\\ 5 & 5 & 4 & 3 & 2 & 1 \end{vmatrix} = +5$$

In reducing this determinant, we have employed the principle of Art. 35. As a still further illustration of the ready application of this principle, we give the steps in the calculation of δ_x.

$$\delta_x = \begin{vmatrix} 5 & 1 & 1 & 1 & 1 & 0 \\ 3 & 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 & 1 & 1 \\ 7 & 1 & 0 & 1 & 1 & 1 \\ 9 & 0 & 1 & 1 & 1 & 1 \\ 11 & 1 & 1 & 1 & 1 & 1 \end{vmatrix} = \begin{vmatrix} 5 & 1 & 1 & 1 & 1 & 0 \\ 2 & 0 & 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 6 & 0 & -1 & 1 & 0 & 1 \\ 8 & -1 & 0 & 1 & 0 & 1 \\ 10 & 0 & 0 & 1 & 0 & 1 \end{vmatrix}$$

$$= -\begin{vmatrix} 5 & 1 & 1 & 1 & 1 \\ 2 & 0 & 0 & 1 & -1 \\ 6 & 0 & -1 & 1 & 0 \\ 8 & -1 & 0 & 1 & 0 \\ 10 & 0 & 0 & 1 & 0 \end{vmatrix} = -\begin{vmatrix} 5 & 1 & 1 & 1 & 1 \\ 7 & 1 & 1 & 2 & 0 \\ 6 & 0 & -1 & 1 & 0 \\ 8 & -1 & 0 & 1 & 0 \\ 10 & 0 & 0 & 1 & 0 \end{vmatrix}$$

$$= -\begin{vmatrix} 7 & 1 & 1 & 2 \\ 6 & 0 & -1 & 1 \\ 8 & -1 & 0 & 1 \\ 10 & 0 & 0 & 1 \end{vmatrix} = -\begin{vmatrix} 7 & 1 & 1 & 2 \\ 13 & 1 & 0 & 3 \\ 8 & -1 & 0 & 1 \\ 10 & 0 & 0 & 1 \end{vmatrix}$$

$$= -\begin{vmatrix} 13 & 1 & 3 \\ 8 & -1 & 1 \\ 10 & 0 & 1 \end{vmatrix} = -\begin{vmatrix} 13 & 1 & 3 \\ 21 & 0 & 4 \\ 10 & 0 & 1 \end{vmatrix} = +\begin{vmatrix} 21 & 4 \\ 10 & 1 \end{vmatrix} = -19.$$

$$\delta_y = \begin{vmatrix} 1 & 5 & 1 & 1 & 1 & 0 \\ 1 & 3 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 & 1 & 1 \\ 1 & 7 & 0 & 1 & 1 & 1 \\ 1 & 9 & 1 & 1 & 1 & 1 \\ 0 & 11 & 1 & 1 & 1 & 1 \end{vmatrix} = -9$$

Similarly

$$\delta_z = +1, \quad \delta_t = +31, \quad \delta_u = +21, \quad \delta_v = +11$$

Therefore we have

$$x = -3\tfrac{3}{5}, \quad y = -1\tfrac{3}{5}, \quad z = \tfrac{1}{5}, \quad t = 6\tfrac{1}{5}, \quad u = 4\tfrac{1}{5}, \quad v = 2\tfrac{1}{5}.$$

2. Solve the system of equations,

$$\begin{aligned} -x_1 + x_2 + x_3 + x_4 &= 8, \\ x_1 - x_2 + x_3 + x_4 &= 6, \\ x_1 + x_2 - x_3 + x_4 &= 4, \\ x_1 + x_2 + x_3 - x_4 &= 2. \end{aligned}$$

3. Solve the simultaneous eqnations

$$\begin{aligned} x - 2y + 3z &= 6, \\ 2x + 3y - 4z &= 20, \\ 3x - 2y + 5z &= 26. \end{aligned}$$

Ans. $x = 8$, $y = 4$, $z = 2$.

42. Number of Eqnations Greater than the Number of Unknowns. In this case where the number of equations in a given system is greater than the number of unknowns, it will not, in general, be possible to solve the system. Whenever values may be assigned to the unknowns which will simultaneously satisfy all the equations, the system is said to be *consistent*. The consistency of any such system must obviously depend upon some relation among the coefficients.

We shall first find what this relation is for the simple case where we have three simultaneous eqnations involving only two unknowns.

Let the given equations be

$$a_1x' + b_1x'' = k_1 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

$$a_2x' + b_2x'' = k_2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

$$a_3x' + b_3x'' = k_3 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

Since the above system is to be consistent, the values of the unknowns obtained by solving any two of the equations must satisfy the third equation.

Solving equations (2) and (3), we get

$$x' = \frac{\begin{vmatrix} k_2 & b_2 \\ k_3 & b_3 \end{vmatrix}}{\begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}} = -\frac{\begin{vmatrix} b_2 & k_2 \\ b_3 & k_3 \end{vmatrix}}{\begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}}; \quad x'' = \frac{\begin{vmatrix} a_2 & k_2 \\ a_3 & k_3 \end{vmatrix}}{\begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}}$$

Substituting these values of x' and x'' in equation (1), and reducing, we get

$$a_1 \begin{vmatrix} b_2 & k_2 \\ b_3 & k_3 \end{vmatrix} - b_1 \begin{vmatrix} a_2 & k_2 \\ a_3 & k_3 \end{vmatrix} + k_1 \begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix} = 0,$$

or

$$\begin{vmatrix} a_1 & b_1 & k_1 \\ a_2 & b_2 & k_2 \\ a_3 & b_3 & k_3 \end{vmatrix} = 0,$$

which is the *condition of consistency* of the three given equations. For example, the system of equations

$$\begin{aligned} 6x' + x'' &= -7, \\ 5x' - 10x'' &= 5, \\ 4x' + 3x'' &= -7 \end{aligned}$$

is consistent, because we have

$$\Delta \equiv \begin{vmatrix} 6 & 1 & -7 \\ 5 & -10 & 5 \\ 4 & 3 & -7 \end{vmatrix} = 0.$$

43. We shall now take up the general case, and investigate this relation in the case of $(n+1)$ linear equations involving n unknowns.

Consider the following system:

$$\left.\begin{array}{l} a_1'x' \;+ a_1''x'' \;+ \cdots + a_1^{(n)}x^{(n)} \;= u_1 \\ a_2'x' \;+ a_2''x'' \;+ \cdots + a_2^{(n)}x^{(n)} \;= u_2 \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ a_n'x' \;+ a_n''x'' \;+ \cdots + a_n^{(n)}x^{(n)} \;= u_n \\ a_{n+1}'x' + a_{n+1}''x'' + \cdots + a_{n+1}^{(n)}x^{(n)} = u_{n+1} \end{array}\right\} \quad . \quad . \quad . \quad (1)$$

Since the above system is to be regarded as consistent, the values of the unknowns obtained by solving any n of the equations must satisfy the remaining equation.

Solving the last n equations by the method of Art. 41, we obtain, after permuting the column, $u_2, u_3 \cdots u_{n+1}$, till it occupies the last position, and having regard to the proper signs:

$$x' = (-1)^{n-1} \frac{\begin{vmatrix} a_2'' & a_2''' & \cdots a_2^{(n)} & u_2 \\ \cdot & \cdot & \cdot & \cdot \\ a_{n+1}'' & a_{n+1}''' & \cdots a_{n+1}^{(n)} & u_{n+1} \end{vmatrix}}{\begin{vmatrix} a_2' & a_2'' & \cdots a_2^{(n)} \\ \cdot & \cdot & \cdot \\ a_{n+1}' & a_{n+1}'' & \cdots a_{n+1}^{(n)} \end{vmatrix}}$$

$$x'' = (-1)^{n-2} \frac{\begin{vmatrix} a_2' & a_2''' & \cdots a_2^{(n)} & u_2 \\ \cdot & \cdot & \cdot & \cdot \\ a_{n+1}' & a_{n+1}''' & \cdots a_{n+1}^{(n)} & u_{n+1} \end{vmatrix}}{\begin{vmatrix} a_2' & a_2'' & \cdots a_2^{(n)} \\ \cdot & \cdot & \cdot \\ a_{n+1}' & a_{n+1}'' & \cdots a_{n+1}^{(n)} \end{vmatrix}}.$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

$$x^{(n)} = \frac{\begin{vmatrix} a_2' & a_2'' & \cdots a_2^{(n-1)} & u_2 \\ \cdot & \cdot & \cdot & \cdot \\ a_{n+1}' & a_{n+1}'' & \cdots a_{n+1}^{(n-1)} & u_{n+1} \end{vmatrix}}{\begin{vmatrix} a_2' & a_2'' & \cdots a_2^{(n)} \\ \cdot & \cdot & \cdot \\ a_{n+1}' & a_{n+1}'' & \cdots a_{n+1}^{(n)} \end{vmatrix}}$$

Substituting these values in the first equation of system (1), clearing of fractions, and reducing, we obtain

$$\begin{vmatrix} a_1' & \cdots & a_1^{(n)} & u_1 \\ a_2' & \cdots & a_2^{(n)} & u_2 \\ \cdot & \cdot & \cdot & \cdot \\ a_n' & \cdots & a_n^{(n)} & u_n \\ a_{n+1}' & \cdots & a_{n+1}^{(n)} & u_{n+1} \end{vmatrix} = 0 \quad . \quad . \quad . \quad . \quad (2)$$

which is the *condition of consistency*, giving the required relation among the coefficients.

When the equations are consistent, this determinant is called the *eliminant* or *resultant* of the system, because it is the result obtained by eliminating the unknowns from the given equations.

We should observe that the resultant, in this case, is the determinant of the *coefficients and absolute terms*.

Example. Test the consistency of the system

$$x + 15y + 14z = 4,$$
$$x + 6y + 7z = 9,$$
$$x + 10y + 11z = 5,$$
$$x + 3y + 2z = 16.$$

Here $$\Delta = \begin{vmatrix} 1 & 15 & 14 & 4 \\ 1 & 6 & 7 & 9 \\ 1 & 10 & 11 & 5 \\ 1 & 3 & 2 & 16 \end{vmatrix} = 0,$$

and the system is consistent.

HOMOGENEOUS LINEAR EQUATIONS.

44. If in equations (1) of the preceding article, the absolute terms (u's) become zeros, we have a system of homogeneous linear equations, and, in this case, the numerators of the frac-

tions giving the values of the unknown quantities vanish. This shows, as we know from other considerations, that such a homogeneous system can always be satisfied by giving to each unknown the value zero. It often happens, however, that such equations may be simultaneously satisfied by assigning to the unknowns values other than zero.

We shall now consider the case of a system of n homogeneous linear equations involving n unknowns.

Let
$$\left.\begin{aligned} a_1'x' + a_1''x'' + \cdots + a_1^{(n)}x^{(n)} &= 0 \\ a_2'x' + a_2''x'' + \cdots + a_2^{(n)}x^{(n)} &= 0 \\ \cdots\cdots\cdots & \\ a_n'x' + a_n''x'' + \cdots + a_n^{(n)}x^{(n)} &= 0 \end{aligned}\right\} \quad . \quad . \quad . \quad (1)$$

be any system of n homogeneous linear equations involving n unknowns $x', x'', \cdots x^{(n)}$, in which the coefficients are so related that

$$\Delta \equiv \begin{vmatrix} a_1' & a_1'' & \cdots & a_1^{(n)} \\ a_2' & a_2'' & \cdots & a_2^{(n)} \\ \cdot & \cdot & \cdot & \cdot \\ a_n' & a_n'' & \cdots & a_n^{(n)} \end{vmatrix} = 0 \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Applying the method of Art. 41 to the system (1), we can obtain the values of the unknowns only in the indeterminate form $\frac{0}{0}$. (Compare Art. 41, Cor.)

Though it is thus impossible to determine the absolute values of the unknowns in such a system as (1), it is possible to find the ratios of any $(n-1)$ of the unknowns to the remaining one.

For, dividing each of the equations (1) by $x^{(i)}$, and representing the ratios

$$\frac{x'}{x^{(i)}}, \frac{x''}{x^{(i)}}, \cdots \frac{x^{(n)}}{x^{(i)}} \text{ by } v', v'', \cdots v^{(n)},$$

respectively, remembering that $v^{(i)} = 1$, we obtain the system

$$\left.\begin{array}{l} a_1'v' + a_1''v'' + \cdots + a_1^{(i-1)}v^{(i-1)} + a_1^{(i+1)}v^{(i+1)} + \cdots + a_1^{(n)}v^{(n)} = -a_1^{(i)} \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ a_n'v' + a_n''v'' + \cdots + a_n^{(i-1)}v^{(i-1)} + a_n^{(i+1)}v^{(i+1)} + \cdots + a_n^{(n)}v^{(n)} = -a_n^{(i)} \end{array}\right\} \quad (3)$$

This is a system of n non-homogeneous linear equations involving $(n-1)$ unknowns, v', v'', $\cdots v^{(i-1)}$, $v^{(i+1)}$, $\cdots v^{(n)}$, and, so long as the condition of consistency (2) holds good, as in Art. 43, the values of the ratios v', v'', $\cdots v^{(i-1)}$, $v^{(i+1)}$, $\cdots v^{(n)}$ may generally be obtained by solving any $n-1$ of equations (3). Hence the system (1) will be satisfied by any values of x', x'', $\cdots x^{(n)}$ among which we have the ratios v', v'', $\cdots v^{(n)}$, as determined by any $n-1$ of equations (3); that is, if equations (1) are satisfied by the values x_0', x_0'', $\cdots x_0^{(n)}$, they will be equally satisfied by $\lambda x_0'$, $\lambda x_0''$, $\cdots \lambda x_0^{(n)}$, λ being any factor.

The relation (2) gives the condition of consistency of the system (1), and Δ, *the determinant of the coefficients*, is the *eliminant* or *resultant* of the system.

To illustrate, let us solve the homogeneous system

$$\left.\begin{array}{l} 2x + 4y + 5z = 0 \\ 3x + 5y + 6z = 0 \\ 4x + 6y + 7z = 0 \end{array}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Here the determinant of the coefficients

$$\Delta \equiv \begin{vmatrix} 2 & 4 & 5 \\ 3 & 5 & 6 \\ 4 & 6 & 7 \end{vmatrix} = 0.$$

An attempt to solve the system (1) by Art. 41 gives

$$x = \frac{0}{0}, \quad y = \frac{0}{0}, \quad z = \frac{0}{0},$$

which are indeterminate. But, since $\Delta = 0$, the system is consistent, and we can obtain definite values for the ratios $\frac{x}{z}$, $\frac{y}{z}$.

Dividing the equations by z, we have

$$\left.\begin{aligned} 2\frac{x}{z}+4\frac{y}{z}&=-5 \\ 3\frac{x}{z}+5\frac{y}{z}&=-6 \\ 4\frac{x}{z}+6\frac{y}{z}&=-7 \end{aligned}\right\} \quad \cdot \cdot \cdot \cdot \cdot \cdot \quad (2)$$

Solving any two of these, we get

$$\frac{x}{z}=+\frac{1}{2},\ \frac{y}{z}=-\frac{3}{2},$$

or $$x:y:z::1:-3:2,$$

and any quantities having these ratios will satisfy the given equations.

45. By the last article the system of homogeneous linear equations

$$\left.\begin{aligned} a_1'x'+a_1''x''+\cdots+a_1^{(n)}x^{(n)}&=0 \\ a_2'x'+a_2''x''+\cdots+a_2^{(n)}x^{(n)}&=0 \\ \cdots\cdots\cdots\cdots\cdots \\ a_n'x'+a_n''x''+\cdots+a_n^{(n)}x^{(n)}&=0 \end{aligned}\right\} \quad \cdot \cdot \cdot \quad (1)$$

is consistent if $\Delta=0$, and in this case we can determine the values of the $(n-1)$ ratios

$$\frac{x'}{x^{(n)}},\ \frac{x''}{x^{(n)}}\cdots\frac{x^{(n-1)}}{x^{(n)}}.$$

Now, since $\Delta=0$, we have (Art. 29) the n equations

$$a_1'A_k'+a_1''A_k''+\cdots+a_1^{(n)}A_k^{(n)}=0,$$

$$\cdots\cdots\cdots\cdots\cdots\cdots$$

$$a_k'A_k'+a_k''A_k''+\cdots+a_k^{(n)}A_k^{(n)}=\Delta=0,$$

$$\cdots\cdots\cdots\cdots\cdots\cdots$$

$$a_n'A_k'+a_n''A_k''+\cdots+a_n^{(n)}A_k^{(n)}=0,$$

which give for the ratios

$$\frac{A_k'}{A_k^{(n)}}, \frac{A_k''}{A_k^{(n)}} \cdots \frac{A_k^{(n-1)}}{A_k^{(n)}},$$

values identical with those which the proposed equations (1) give for the ratios

$$\frac{x'}{x^{(n)}}, \frac{x''}{x^{(n)}} \cdots \frac{x^{(n-1)}}{x^{(n)}};$$

therefore $A_k', A_k'' \cdots A_k^{(n)}$ are proportional to $x', x'' \cdots x^{(n)}$ whatever may be the index k, so that we have the proportions

$$\begin{aligned} x' : x'' : \cdots : x^{(n)} &= A_1' : A_1'' : \cdots : A_1^{(n)} \\ &= A_2' : A_2'' : \cdots : A_2^{(n)} \\ &\cdots\cdots\cdots \\ &= A_n' : A_n'' : \cdots : A_n^{(n)}. \end{aligned}$$

Hence, *in any determinant which equals zero, the minors of the elements in any row (or column) are proportional to the minors of the corresponding elements in any other row (or column).*

46. Among the proportions of Art. 45, let us consider those of the last line, for example

$$x' : x'' : \cdots : x^{(n)} = A_n' : A_n'' : \cdots : A_n^{(n)}. \quad . \quad . \quad . \quad (1)$$

The coefficients of the last of the proposed equations ((1) of Art. 45)

$$a_n'x' + a_n''x'' + \cdots + a_n^{(n)}x^{(n)} = 0,$$

not appearing in the expressions for $A_n', A_n'', \cdots, A_n^{(n)}$, there results that the proportions (1) determine the ratios of the unknowns $x', x'', \cdots, x^{(n)}$, which will satisfy the $n-1$ equations

$$\begin{aligned} a_1'x' + a_1''x'' + \cdots + a_1^{(n)}x^{(n)} &= 0 \\ \cdots\cdots\cdots& \\ a_{n-1}'x' + a_{n-1}''x'' + \cdots + a_{n-1}^{(n)}x^{(n)} &= 0 \end{aligned}$$

expressed by means of the minors which can be formed with the $n(n-1)$ coefficients of these equations, in suppressing in turn each of the vertical lines. Therefore having given n homogeneous equations between $n+1$ unknowns

$$a_1'x' + a_1''x'' + \cdots + a_1^{(n+1)}x^{(n+1)} = 0$$

$$\cdots\cdots\cdots\cdots$$

$$a_n'x' + a_n''x'' + \cdots + a_n^{(n+1)}x^{(n+1)} = 0$$

if we put

$$R^{(i)} = (-1)^{i-1}\begin{vmatrix} a_1' \cdots a_1^{(i-1)} & a_1^{(i+1)} \cdots a_1^{(n+1)} \\ \cdots\cdots & \cdots\cdots \\ a_n' \cdots a_n^{(i-1)} & a_n^{(i+1)} \cdots a_n^{(n+1)} \end{vmatrix}$$

the solution of the proposed equations would be given by the proportions

$$x' : x'' : \cdots : x^{(n+1)} = R' : R'' : \cdots : R^{(n+1)}.$$

For example, the two equations

$$\left.\begin{array}{r} -4x + y + z = 0 \\ x - 2y + z = 0 \end{array}\right\}$$

give

$$x : y : z = \begin{vmatrix} 1 & 1 \\ -2 & 1 \end{vmatrix} : - \begin{vmatrix} -4 & 1 \\ 1 & 1 \end{vmatrix} : \begin{vmatrix} -4 & 1 \\ 1 & -2 \end{vmatrix}$$

$$= 3 : 5 : 7.$$

§ 7 DETERMINANTS OF SPECIAL FORMS.

47. Symmetrical Determinants. Two elements of a determinant so situated, that one occupies with reference to the leading element the same position in the rows as the other does in the columns, are called *conjugate* elements. For example, in the common form of determinant, d_2 and b_4 are conjugates, one occupying the fourth place in the second row, and the other the fourth place in the second column.

Each of the leading elements (that is, the elements of the *principal diagonal*) is its own conjugate. Any two conjugate elements are situated in a line perpendicular to the principal diagonal, and at equal distances from it on opposite sides.

A *symmetrical* determinant is one in which each element has itself for a conjugate element. Examples of symmetrical determinants are the following:

$$\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} \quad (1), \qquad \begin{vmatrix} 0 & x & y & z \\ x & 0 & z & y \\ y & z & 0 & x \\ z & y & x & 0 \end{vmatrix} \quad (2)$$

In a symmetrical determinant the first minors complementary to any two conjugate elements are equal, since they differ only by an interchange of rows and columns. The corresponding inverse elements are also equal, the signs to be attached to the minors being the same in both cases. It follows that the *reciprocal of a symmetrical determinant is itself symmetrical.*

The leading minors are all symmetrical determinants.

The principal diagonal is called the *axis of symmetry.*

EXAMPLES.

1. Form the reciprocal of the symmetrical determinant

$$\Delta \equiv \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}$$

Using the capital letters to denote the reciprocal elements (Art. 39), the reciprocal determinant may be written thus:

$$\Delta' \equiv \begin{vmatrix} A & H & G \\ H & B & F \\ G & F & C \end{vmatrix} \equiv \begin{vmatrix} bc-f^2 & fg-ch & hf-bg \\ fg-ch & ca-g^2 & gh-af \\ hf-bg & gh-af & ab-h^2 \end{vmatrix}$$

2. Prove by means of the proposition of Art. 36. that the square of any determinant is a symmetrical determinant.

48. Skew-Symmetric and Skew Determinants. A *skew-symmetric* determinant is one in which each element is its conjugate with sign changed. Since each leading element is its own conjugate, it follows that in such a determinant all the elements of the principal diagonal are zero. For example, the determinant

$$\Delta \equiv \begin{vmatrix} 0 & a & b & c \\ -a & 0 & d & e \\ -b & -d & 0 & f \\ -c & -e & -f & 0 \end{vmatrix}$$

is skew-symmetric.

A *skew* determinant is one in which each element, *except the leading elements*, is its conjugate with sign changed.

Thus, while a skew-symmetric determinant is zero-axial, a skew determinant is not. Thus

$$\Delta \equiv \begin{vmatrix} x & a & b & c \\ -a & y & l & m \\ -b & -l & z & n \\ -c & -m & -n & w \end{vmatrix}$$

is a skew determinant.

MISCELLANEOUS EXAMPLES.

Evaluate the following determinants:

1. $\begin{vmatrix} 2 & 1 & 10 \\ 3 & 0 & 6 \\ 4 & 5 & 7 \end{vmatrix}$ 2. $\begin{vmatrix} 1 & 3 & 0 \\ 2 & 0 & 5 \\ 4 & 6 & 7 \end{vmatrix}$ 3. $\begin{vmatrix} 25 & 5 & 10 \\ 15 & 3 & 9 \\ 1 & 2 & 3 \end{vmatrix}$

4. $\begin{vmatrix} 10 & 4 & 5 \\ 15 & 5 & 6 \\ 20 & 6 & 8 \end{vmatrix}$ 5. $\begin{vmatrix} 10 & 0 & 5 \\ 15 & 3 & 6 \\ 20 & 4 & 7 \end{vmatrix}$ 6. $\begin{vmatrix} 1 & 3 & 4 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{vmatrix}$

7. $\begin{vmatrix} 1 & \frac{1}{2} & \frac{1}{3} \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{4} \\ \frac{1}{3} & \frac{1}{4} & \frac{1}{5} \end{vmatrix}$ 8. $\begin{vmatrix} 4 & 5 & 2 \\ -1 & 2 & -3 \\ 6 & -4 & 5 \end{vmatrix}$ 9. $\begin{vmatrix} 1 & -1 & 1 \\ 4 & -3 & 0 \\ 3 & 2 & -5 \end{vmatrix}$

10. $\begin{vmatrix} 4 & -1 & -2 \\ 0 & 3 & 0 \\ 3 & -7 & 4 \end{vmatrix}$ 11. $\begin{vmatrix} -1 & -1 & 1 \\ -3 & 1 & -4 \\ 2 & -3 & -5 \end{vmatrix}$ 12. $\begin{vmatrix} 15 & 17 & 16 \\ 12 & 18 & 14 \\ 19 & 17 & 13 \end{vmatrix}$

13. $\begin{vmatrix} 15 & 13 & 10 \\ 12 & 17 & 10 \\ 16 & 11 & 19 \end{vmatrix}$ 14. $\begin{vmatrix} 20 & 15 & 25 \\ 17 & 12 & 22 \\ 19 & 20 & 16 \end{vmatrix}$ 15. $\begin{vmatrix} 30 & 36 & 35 \\ 33 & 31 & 37 \\ 38 & 34 & 32 \end{vmatrix}$

16. Expand and simplify the determinant

$$\begin{vmatrix} a & a+3 & a+6 \\ a+1 & a+4 & a+7 \\ a+2 & a+5 & a+8 \end{vmatrix}$$ *Ans.* 0.

Evaluate the following determinants:

17. $\begin{vmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & -1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \end{vmatrix}$ *Ans.* $+16$.

18. $\begin{vmatrix} 2 & 3 & -1 & 5 \\ 0 & 6 & -5 & -3 \\ 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \end{vmatrix}$ *Ans.* -74.

19. $\begin{vmatrix} 2 & 2 & 2 & 10 \\ 0 & 0 & 1 & 2 \\ 3 & 4 & -3 & 2 \\ 1 & -1 & 4 & 5 \end{vmatrix}$ *Ans.* $+16$.

20. $\begin{vmatrix} 6 & 3 & 2 & 1 \\ 5 & 8 & 7 & 2 \\ 4 & 2 & 8 & 4 \\ 3 & 6 & 3 & 3 \end{vmatrix}$ *Ans.* $+660$.

21. $\begin{vmatrix} 1 & 3 & 5 & 2 \\ 1 & 2 & 5 & 1 \\ 2 & 4 & 4 & 3 \\ 5 & 2 & 2 & 1 \end{vmatrix}$

22. $\begin{vmatrix} 0 & 0 & 0 & 4 \\ 0 & 0 & 1 & 5 \\ 0 & 3 & 2 & 4 \\ 6 & 2 & 0 & 3 \end{vmatrix}$

23. $\begin{vmatrix} 3 & 1 & 4 & 1 \\ 2 & 2 & 8 & 5 \\ 1 & 6 & 4 & 2 \\ 3 & 2 & 5 & 3 \end{vmatrix}$

Ans. -101.

24. $\begin{vmatrix} 3 & 7 & 4 & 3 \\ 7 & 4 & 3 & 5 \\ 2 & 1 & 9 & 4 \\ 8 & 6 & 4 & 7 \end{vmatrix}$

Ans. -336.

25. $\begin{vmatrix} 2 & 1 & 1 & 2 \\ 0 & 3 & 4 & 0 \\ 5 & 2 & 2 & 5 \\ 0 & 0 & 7 & 5 \end{vmatrix}$

26. $\begin{vmatrix} 10 & 8 & 9 & 14 \\ 17 & 15 & 18 & 11 \\ 15 & 19 & 10 & 13 \\ 16 & 17 & 18 & 10 \end{vmatrix}$

Ans. -2660.

27. $\begin{vmatrix} 3 & 1 & 5 & 4 & 2 \\ 7 & 6 & 4 & 1 & 3 \\ 1 & 3 & 2 & 9 & 4 \\ 2 & 2 & 9 & 2 & 1 \\ 8 & 6 & 1 & 3 & 4 \end{vmatrix}$

Ans. 172.

28. $\begin{vmatrix} 5 & -1 & 4 & 6 & -2 \\ -1 & 4 & 6 & -2 & 5 \\ 4 & 6 & -2 & 5 & -1 \\ 6 & -2 & 5 & -1 & 4 \\ -2 & 5 & -1 & 4 & 6 \end{vmatrix}$

Ans. $+22,692$.

29. $\begin{vmatrix} 3 & 4 & 7 & 2 & 5 \\ -3 & 1 & 2 & 5 & -1 \\ 6 & -2 & 3 & -1 & 4 \\ 5 & 9 & -2 & 3 & 2 \\ 1 & -3 & 5 & 3 & 7 \end{vmatrix}$

30. $\begin{vmatrix} 0 & 3 & 7 & 9 & 5 \\ -3 & 0 & 4 & 2 & 1 \\ -7 & -4 & 0 & 11 & 101 \\ -9 & -2 & -11 & 0 & 2 \\ -5 & -1 & -101 & -2 & 0 \end{vmatrix}$

31. $\begin{vmatrix} 1 & 1 & 5 & 1 & 1 & 0 \\ 1 & 1 & 3 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 7 & 1 & 1 & 1 \\ 1 & 0 & 9 & 1 & 1 & 1 \\ 0 & 1 & 11 & 1 & 1 & 1 \end{vmatrix}$

Ans. $\cancel{0}$!

32. $\begin{vmatrix} 2 & 4 & 3 & 1 & 4 & 3 \\ -4 & 2 & -3 & 2 & -1 & 2 \\ 5 & -1 & 6 & 2 & -1 & 5 \\ 1 & 1 & 1 & -2 & -2 & -2 \\ 7 & -3 & -5 & 1 & 4 & 2 \\ 3 & 1 & 2 & -1 & 2 & 3 \end{vmatrix}$

Ans. $+14,940$.

33. $$\begin{vmatrix} 12 & 22 & 14 & 17 & 20 & 10 \\ 16 & -4 & 7 & 1 & -2 & 15 \\ 10 & -3 & -2 & 3 & -2 & 8 \\ 7 & 12 & 8 & 9 & 11 & 6 \\ 11 & 2 & 4 & -8 & 1 & 9 \\ 24 & 6 & 6 & 3 & 4 & 22 \end{vmatrix}$$ *Ans.* 12,228.

34. Find the number of *inversions* in the series

$$b\ a\ c\ f\ i\ g\ d\ h\ e.$$

35. Find the number of inversions in the following permutations:

3, 6, 4, 1, 5, 2;
7, 1, 6, 5, 3, 4, 2;
2, 4, 1, 3, 6, 7, 5;
4, 8, 6, 7, 2, 5, 3;
3, 1, 8, 9, 2, 5, 6, 7, 4.

Develop the following determinants:

36. $$\begin{vmatrix} x & 0 & y \\ 0 & x & y \\ -x & -y & 0 \end{vmatrix}$$

37. $$\begin{vmatrix} a & b & b \\ b & a & b \\ b & b & a \end{vmatrix}$$

38. $$\begin{vmatrix} 0 & d & d & d \\ a & 0 & a & a \\ b & b & 0 & b \\ c & c & c & 0 \end{vmatrix}$$

39. $$\begin{vmatrix} a & 1 & 0 & 0 \\ -1 & b & 1 & 0 \\ 0 & -1 & c & 1 \\ 0 & 0 & -1 & d \end{vmatrix}$$

Ans. $abcd + ab + ad + cd + 1$.

40. $$\begin{vmatrix} 1+a & 1 & 1 & 1 \\ 1 & 1+b & 1 & 1 \\ 1 & 1 & 1+c & 1 \\ 1 & 1 & 1 & 1+d \end{vmatrix}$$

Ans. $abcd(1 + a^{-1} + b^{-1} + c^{-1} + d^{-1})$.

41. $\begin{vmatrix} 0 & 0 & \bar{k} & l & x \\ 0 & 0 & h & x & 0 \\ 0 & 0 & x & 0 & 0 \\ 0 & x & e & f & g \\ x & a & b & c & d \end{vmatrix}$ *Ans.* x^5.

42. $\begin{vmatrix} a_1 & a_2 & a_3 & a_4 \\ 0 & b_2 & 0 & 0 \\ 0 & c_2 & c_3 & c_4 \\ 0 & d_2 & 0 & d_4 \end{vmatrix}$ *Ans.* $a_1b_2c_3d_4$.

43. Write the expanded form of the determinants:

$$| a_1'a_2''a_3''' |;$$
$$| a_{11}a_{22}a_{33}a_{44}a_{55} ;$$
$$\Sigma \pm a_1b_2c_3d_4e_5.$$

Find the values of x in the following equations:

44. $\begin{vmatrix} x-4 & 1 \\ -6 & 3-2 \\ x & 2 & 1 \end{vmatrix} = 0.$

45. $\begin{vmatrix} 1 & 1 & 1 \\ a & x & c \\ b & b & x \end{vmatrix} = 0.$

46. $\begin{vmatrix} a+bx & c & d \\ e+fx & g & h \\ i+kx & l & m \end{vmatrix} = 0.$ *Ans.* $x = -\dfrac{|agm|}{|bgm|}$.

47. $\begin{vmatrix} 0 & x & 3 \\ 1-x & 4 \\ 2 & 5-6 \end{vmatrix} = 0.$ *Ans.* $x = -\frac{3}{4}$.

48. What effect is produced on a determinant of the nth degree by multiplying all its elements by -1?

49. Show that $\begin{vmatrix} 1 & 1 & 1 & 1 \\ 1 & 1+x & 1 & 1 \\ 1 & 1 & 1+y & 1 \\ 1 & 1 & 1 & 1+z \end{vmatrix} = xyz.$

50. Show that

$$\begin{vmatrix} 0 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = \begin{vmatrix} 0 & 1 & 1 \\ 1 & a_3b_2c_1 & a_2b_3c_1 \\ 1 & a_3b_1c_2 & a_2b_1c_3 \end{vmatrix}$$

51. Prove that

$$\begin{vmatrix} a+c & b+d & a+c & b+d \\ b+d & a+c & b+d & a+c \\ a+b & b+c & c+d & d+a \\ c+d & d+a & a+b & b+c \end{vmatrix} = 0.$$

52. Show that

$$\begin{vmatrix} 0 & a & b & c \\ a & 0 & c & b \\ b & c & 0 & a \\ c & b & a & 0 \end{vmatrix} = \begin{vmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & c^2 & b^2 \\ 1 & c^2 & 0 & a^2 \\ 1 & b^2 & a^2 & 0 \end{vmatrix}.$$

Resolve into simple factors the two determinants:

53. $\begin{vmatrix} x & 1 & 1 & 1 \\ 1 & x & 1 & 1 \\ 1 & 1 & x & 1 \\ 1 & 1 & 1 & x \end{vmatrix}$ *Ans.* $(x+3)(x-1)^3$.

54. $\begin{vmatrix} a & a & a & a \\ a & b & b & b \\ a & b & c & c \\ a & b & c & d \end{vmatrix}$ *Ans.* $-a(a-b)(b-c)(c-d)$.

55. Transform $\begin{vmatrix} a & b & c & d \\ b & c & d & a \\ c & d & a & b \\ d & a & b & c \end{vmatrix}$

so as to have the principal diagonal composed (1) of the four a's, (2) of the four b's, (3) of the four c's, (4) of the four d's.

Prove the following identities:

56. $\begin{vmatrix} a+b & c & c \\ a & b+c & a \\ b & b & c+a \end{vmatrix} \equiv 4abc.$

57. $$\begin{vmatrix} \dfrac{a^2+b^2}{c} & c & c \\ a & \dfrac{b^2+c^2}{a} & a \\ b & b & \dfrac{c^2+a^2}{b} \end{vmatrix} \equiv 4abc.$$

58. $$\begin{vmatrix} a+b-c & c & c \\ a & b+c-a & a \\ b & b & c+a-b \end{vmatrix} \equiv \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}$$

59. Find the value of

$$\begin{vmatrix} 1 & 2 & 1 & 1 \\ 3 & 0 & 1 & 4 \\ 0 & 2 & 1 & -1 \\ 2 & 3 & 0 & -4 \end{vmatrix} \times \begin{vmatrix} -1 & 4 & 2 & -1 \\ 2 & -1 & 3 & -2 \\ 0 & 2 & -1 & 1 \\ 3 & 0 & 4 & -1 \end{vmatrix}$$

Perform the following multiplications, giving the results as determinants:

60. $$\begin{vmatrix} a & b & 0 \\ c & 0 & d \\ 0 & e & f \end{vmatrix} \cdot \begin{vmatrix} 0 & a & b \\ c & 0 & d \\ e & f & 0 \end{vmatrix}$$

Ans. $$\begin{vmatrix} ab & ac & ae+bf \\ bd & c^2+d^2 & ce \\ ae+bf & df & ef \end{vmatrix} \text{ etc.}$$

61. $$\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} \cdot \begin{vmatrix} a^2 & -a & 1 \\ b^2 & -b & 1 \\ c^2 & -c & 1 \end{vmatrix}$$

Ans. $$\begin{vmatrix} a^2 & a^2-ab+b^2 & a^2-ac+c^2 \\ a^2-ab+b^2 & b^2 & b^2-bc+c^2 \\ a^2-ac+c^2 & b^2-bc+c^2 & c^2 \end{vmatrix}$$

62. $\begin{vmatrix} a & a & a & a \\ a & b & b & b \\ a & b & c & c \\ a & b & c & d \end{vmatrix} \cdot \begin{vmatrix} -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \\ 1 & 1 & 1 & -1 \end{vmatrix}$.

Solve, by means of determinants, the following equations:

63. $\left.\begin{aligned} 3x+5y&=17 \\ 2x+3y&=11 \end{aligned}\right\}$

64. $\left.\begin{aligned} -4x+7y-10&=0 \\ 7x-4y+1&=0 \end{aligned}\right\}$

65. $\left.\begin{aligned} 3x-4y+2z&=1 \\ 2x+3y-3z&=-1 \\ 5x-5y+4z&=7 \end{aligned}\right\}$ *Ans.* 1, 2, 3.

66. $\left.\begin{aligned} 4x-7y+z&=16 \\ 3x+y-2z&=10 \\ 5x-6y-3z&=10 \end{aligned}\right\}$ *Ans.* 5, 1, 3.

67. $\left.\begin{aligned} 5x-4z&=42 \\ 3z+5y&=1 \\ 4y-3x&=-10 \end{aligned}\right\}$ *Ans.* 6, 2, −3.

68. $\left.\begin{aligned} 4x+7y+3z-2w&=9 \\ 2x-y-4z+3w&=13 \\ 3x+2y-7z-4w&=2 \\ 5x-3y+z+5w&=13 \end{aligned}\right\}$ *Ans.* 1, 3, −1, 3.

69. $\left.\begin{aligned} 3x+2y+4z-w&=13 \\ 5x+y-z+2w&=9 \\ 2x+3y-7z+3w&=14 \\ 4x-4y+3z-5w&=4 \end{aligned}\right\}$ *Ans.* 2, 4, −1, −3.

70. What relation must exist between a, b, c, d if the equations

$$\begin{aligned} ax+by+cz+d &= 0, \\ bx+ay+dz+c &= 0, \\ ax+cy+bz+d &= 0, \\ cx+ay+dz+b &= 0, \end{aligned}$$

be simultaneously true?

71. Test the consistency of the system

$$\left.\begin{aligned} x+10y+14z &= 3 \\ 2x-\ 6y+\ 7z &= 8 \\ x+12y+11z &= 4 \\ x-\ 3y+\ 2z &= 12 \end{aligned}\right\}$$

72. Test the consistency of the system

$$\left.\begin{aligned} 2x-3y+10z &= 4 \\ x+4y-\ 8z &= 2 \\ 3x+\ y+\ 2z &= 6 \\ 4x+5y+\ z &= 8 \end{aligned}\right\}$$

73. Solve the homogeneous equations

$$\left.\begin{aligned} x+2y+3z &= 0 \\ 2x+3y+4z &= 0 \\ 3x+4y+5z &= 0 \end{aligned}\right\}$$

74. A skew-symmetric determinant of odd order vanishes.

For any skew-symmetric determinant, Δ (see Art. 48) is unaltered by changing the columns into rows, and then changing the signs of all the rows. But when the order of the determinant is odd, this process ought to change the sign of Δ; hence Δ must in this case vanish. For example,

$$\Delta \equiv \begin{vmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{vmatrix} \equiv 0$$

We give as our last example a special *determinant as the product of differences.* Exercises 7 and 8, after Art. 21, have afforded examples of the resolution of this particular form of a determinant, of which we now consider the general case.

75. Take any n quantities, $a, b, c, \cdots, k, l$, and form a determinant containing as rows (or columns) the powers of these quantities from 0 to $n-1$; thus:

$$\Delta \equiv \begin{vmatrix} 1 & 1 & 1 & \cdots & 1 & 1 \\ a & b & c & \cdots & k & l \\ a^2 & b^2 & c^2 & \cdots & k^2 & l^2 \\ a^3 & b^3 & c^3 & \cdots & k^3 & l^3 \\ \cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\ a^{n-1} & b^{n-1} & c^{n-1} & \cdots & k^{n-1} & l^{n-1} \end{vmatrix}$$

This determinant possesses the property of vanishing when any two of the n numbers are equal, for example, if we put:

$$a=b,\ a=c,\ \cdots,\ a=l,\ b=c,\ b=d, \text{ etc.},$$

since then two columns become identical. It results that Δ ought to contain as factors all the differences which can be formed with the series

$$a,\ b,\ c,\ \cdots\ k,\ l,$$

in subtracting from each letter all the letters that follow it. The product P of these differences would be

$$\begin{aligned} P = (a-b)(a-c)(a-d)\cdots(a-k)(a-l)& \\ (b-c)(b-d)\cdots(b-k)(b-l)& \\ (c-d)\cdots(c-k)(c-l)& \\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot& \\ (h-k)(h-l)& \\ (k-l).& \end{aligned}$$

The determinant Δ is equal to P in absolute value. For the degree of Δ with respect to $a, b, c, \cdots, l$, is equal to

$$1+2+3+\cdots+(n-1)=\frac{n(n-1)}{1\cdot 2},$$

as we see from its principal term; this is also the degree of P, which embraces $\frac{n(n-1)}{1\cdot 2}$ differences; therefore P and Δ can differ from each other only by a numerical factor. Finally, to determine this factor, we remark that the principal term of Δ is the expression

$$1\cdot b\cdot c^2\cdot d^3\cdots k^{n-2}\cdot l^{n-1}.$$

The corresponding term of the product P, obtained in considering the columns, we find to be

$$(-1)b\cdot(-1)^2c^2\cdot(-1)^3d^3\cdots(-1)^{n-2}k^{n-2}\cdot(-1)^{n-1}l^{n-1}.$$

This has for coefficient

$$(-1)^{1+2+3+\cdots+(n-1)}=(-1)^{\frac{n(n-1)}{2}}.$$

Therefore, we have

$$\Delta=\pm P,$$

according as $\frac{n(n-1)}{2}$ is even or odd.

PART II. — THEORY OF EQUATIONS.

INTRODUCTION.

Historical Note. While we cannot, in this brief notice, go back to the beginnings of algebra, a few historical notes may prove of interest to the reader.* The first comprehensive algebra was published in 1494 by Lucas Pacioli, an Italian mathematician. Scipio Ferro (Professor of Mathematics at Bologna from 1496 to 1525) first solved a cubic equation of the form $x^3 + mx = n$. His method is not known.

A second solution of cubics was given by Nicolo, called Tartaglia (1506–1557). This solution, known as *Cardan's Solution*, was stolen by Hieronimo Cardano (1501–1576) and published in 1545 in Cardan's *Ars Magna*. Ferrari (a pupil of Cardan's) discovered a general solution of bi-quadratic equations, which was also published in the *Ars Magna*, a work far in advance of any algebra previously printed. About the middle of the sixteenth century negative roots were receiving considerable attention, but it seems impossible to say who first fully comprehended them. Bombelli, in his algebra published in 1572, opened the way to the recognition of imaginary roots. Here, too, progress was slow. Michael Stifel was the greatest German algebraist of the sixteenth century. Vieta (1540–1603), the most eminent French mathematician of the sixteenth century, enriched algebra by innovations in notation, and by numerous discoveries in the Theory of Equations. Thomas Harriot (1560–1621), of England, made further improvement in notation, and did much to establish the Theory of Equations on a scientific basis. After this it was enriched by the fruitful discoveries of Descartes, Newton, Lagrange, Argand, Gauss, Abel, Hermite, Kronecker, Cayley, Sylvester, and others. The solution of numerical equations was particularly advanced by Fourier, Budan, Horner, and Sturm.

There are many text-books in which the subject is discussed, among them we may mention: Burnside and Panton's *Theory of Equations*; Todhunter's

* An excellent history of mathematics, and perhaps the one most easily accessible to the reader, is Cajori's *A History of Mathematics*, Macmillan & Co. Interesting historical notes may be found in Fine's *The Number-System of Algebra*.

An Elementary Treatise on the Theory of Equations; Serret's *Cours d'Algèbre Supérieure*; Carnoy's *Cours d'Algèbre Supérieure*; Biermann's *Elemente der Höheren Mathematik*; Matthiessen's *Grundzüge der Antiken und Modernen Algebra der Litteralen Gleichungen*;* Petersen's *Algebraische Gleichungen*.

49. In elementary algebra the student has solved equations of the first and second degrees, and has become somewhat familiar with the meaning of the word *root* as applied to an equation; and some of the definitions given in these pages, as well as some of the processes described and employed, will not be entirely new to him. Let us consider the theorem:

An integral equation of the first degree in one unknown has one and only one solution.

For example, take the equation

$$ax + b = 0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

One solution of this is $x = -\frac{b}{a}$. To prove that this is the only root, let us suppose that there are two distinct solutions, $x = \alpha$, and $x = \beta$, of (1). Then we must have

$$a\alpha + b = 0,$$

$$a\beta + b = 0.$$

From these, by subtraction, we derive

$$a(\alpha - \beta) = 0.$$

Now, by hypothesis, a is not $= 0$, therefore we must have $\alpha - \beta = 0$, that is, $\alpha = \beta$; in other words, the two solutions are not distinct. Hence there is only one root, and it is a function of the coefficients.†

* Matthiessen develops the subject historically, and on pages 964–1001 may be found a very extended bibliographical list.

† As is well known, the constant term b is called a coefficient, and it is the coefficient of x^0.

The quadratic eqnation

$$ax^2+bx+c=0 \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

has two roots, namely,

$$\frac{-b+\sqrt{b^2-4ac}}{2a}, \text{ and } \frac{-b-\sqrt{b^2-4ac}}{2a};$$

and with respect to these roots, we know that their sum is $-\frac{b}{a}$, and their product is $\frac{c}{a}$; that is, their snm is equal to the coefficient of the second term of the equation

$$x^2+\frac{b}{a}x+\frac{c}{a}=0,$$

with its sign changed, and their prodnct is equal to the last term of this equation. Thns the stndent has seen that the root of an equation of the first or second degree may be expressed in terms of its coefficients.

The general object of this treatise is to establish results with respect to equations of a higher degree than the second, similar to those that have been established in elementary algebra respecting equations of the second degree. In fact, the science of the Theory of Equations seeks to discover general methods for the solution of equations of any degree. The limitations to this search will appear later (see Art. 53).

50. Definitions. Any algebraic expression that depends npon any quantity as x for its value is said to be a *function* of x. Thus $3x^2-4x+16$ is a function of x, so also is $\sqrt{a^2-x^2}$.

An *algebraic function* involves the operations of addition, subtraction, multiplication, and division applied only a finite number of times.* All other functions are called *transcendental functions*, such as *logarithmic*, *exponential*, *trigonometric*, and *in-*

* This of course includes involution and evolution with constant exponents. See Appendix A.

verse trigonometric. In this work, when we use the word *function*, we mean an algebraic function, unless it is expressly stated or shown by the form that the function is transcendental.

A function of x is, for brevity, represented by $F(x)$, $f(x)$, $\phi(x)$, or some such symbol. Thus, for example,

$$F(x) \equiv 3x^2 - 4x + 16, \quad f(x) \equiv a \log x, \quad \phi(x) \equiv \sin 3x.$$

A *rational function* of a quantity is one that contains the quantity in a rational form only; that is, a form free from fractional indices or radical signs.

An *integral function* of a quantity is a rational function in which the quantity enters in an integral form only; that is, never in the denominator of a fraction.

A *rational integral function* of x, as discussed here, is one that can be put in the form

$$ax^n + bx^{n-1} + cx^{n-2} + \cdots + kx + l,$$

in which n is a positive whole number, and $a, b, c \cdots l$ denote any real expressions not containing x. It will be observed that the coefficients may be irrational or fractional.

Algebraic symbols are numerals, letters of the alphabet, or conventional signs to denote certain operations or relations, such as $-$, $+$, $\times$, $\div$, $=$, $>$, or $<$, etc.

An *algebraic expression* is any combination of algebraic symbols which represents a quantity.

A *term* is an expression whose parts are not separated by the signs $+$ or $-$, as $4x^3$, $3abc$, or $\frac{m}{n}$.

A *monomial* is an algebraic expression of one term; a *polynomial* is one of two or more terms.

51. An *identical equation* is the statement of equality between mathematical expressions which are either the same, initially, or become the same by the application to one or both of the allowable mathematical operations; for example,

$$x^2 - y^2 \equiv (x - y)(x + y), \quad \sin 2A \equiv 2 \sin A \cos A,$$

are identical equations.

If one algebraic expression containing x is equal, for certain values of x, to another differently constituted, the equality thus formed is called an *equation of condition*. Whenever an *equation of condition* is meant, we shall use the single word *equation*.

An *equation*, then, is the statement of an equality, which is true only for certain values of the unknown quantity.

Any value of x which satisfies this equation is called a *root* of the equation. The determination of all possible roots constitutes the complete *solution of the equation*.

By bringing all the terms to one side, we may obviously arrange any equation according to descending powers of x in the following way:

$$a_0x^n + a_1x^{n-1} + a_2x^{n-2} + a_3x^{n-3} + \cdots + a_{n-1}x + a_n = 0 \quad . \quad (1)$$

An equation is not altered if all of its terms be divided by any quantity. Dividing (1) by a_0, and thus making the coefficient of x^n equal to unity, it may be written in the form:

$$x^n + p_1x^{n-1} + p_2x^{n-2} + \cdots + p_{n-1}x + p_n = 0 \quad . \quad . \quad (2)$$

The highest power of x in this equation being n, it is said to be an equation of the *nth degree* in x.

An equation is *complete* when it contains terms involving x in all its powers from n to 0, and *incomplete* when some of the terms are absent; that is, when some of the coefficients, a_1, a_2, a_3 etc., are equal to zero.

The term a_n, which does not contain x, is called the *absolute term*.

52. A *numerical equation* is an equation in which the coefficients are represented by figures only; a *literal equation* is one in which the coefficients are represented wholly or in part by letters.

A *linear equation* is one of the first degree.
A *quadratic equation* is one of the second degree.
A *cubic equation* is one of the third degree.
A *biquadratic*, or *quartic equation* is one of the fourth degree.
A *quintic equation* is one of the fifth degree.
A *sextic equation* is one of the sixth degree.
Equations above the second degree are called *higher equations.*

53. In both mathematical and physical researches, we frequently meet with problems that involve the solution of equations.

As the equations thus met with are often higher than the second degree, it becomes a matter of importance to find, if possible, some general method for the solution of higher equations. In the case where the coefficients of an equation are given numbers, very great progress has been made in discovering methods for the determination of the numerical values of the roots; but the same progress has not been made in the general solution of equations whose coefficients are letters.

We have seen (Art. 49) that there is a general algebraic solution of literal equations of the second degree. Similar formulas (subject to some limitations) have been discovered for the solution of equations of the third and fourth degrees.

Many attempts were made to reduce similar general formulas for equations of the fifth and higher degrees, but without success; and, finally, in 1824, Abel* proved the impossibility of solving by radicals an algebraic equation of the fifth degree, or, in general, of any degree higher than the fourth. This important proof was published by Abel in 1826.† In modern form it may be be found in Biermann.‡ Serret § gives a simpler proof by Wantzel.

* Niels Henrick Abel (1802–1829), of Norway.

† *Mémoire sur les Equations Algébriques*: Christiania, 1826. Also in *Crelle's Journal*, Bd. I., 1826.

‡ *Elemente der Höheren Mathematik.* § *Cours d'Algèbre Supérieure*, Tome II

CHAPTER IV

COMPLEX NUMBERS.

54. In the solution of quadratic equations, the student has frequently met with the square root of a negative quantity. Such a number is said to be *imaginary* or *unreal*, for the square of no real quantity is negative. The imaginary unit $\sqrt{-1}$ is denoted for brevity by i, and integral powers of i beyond the first can always be reduced by the relation $i^2 = -1$. All the operations that we perform on the unit i must, then, be subject to this definition, $i^2 = -1$, and to the general laws of algebra. For example, $yi = iy$, $yi + y'i = (y + y')i = i(y + y')$, etc., exactly as if i were a real quantity.

55. If we combine, by addition, any real quantity a with a purely imaginary quantity bi, there arises a mixed quantity $a + bi$, a form frequently met with.

Such an expression, consisting of a positive or negative real units and b positive or negative imaginary units, is called a *complex* number, or quantity. (Throughout this book we make no distinction between the words "number" and "quantity.")

Real and purely imaginary numbers are both included in the expression $a + ib$, the former being obtained when $b = 0$, and the latter when $a = 0$.

Of course, in such expressions, a and b are considered *real*.

56. The successive powers of i are periodic. We have:

$$i^1 = i, \qquad i^2 = -1, \qquad i^3 = i^2 \cdot i = -i,$$

$$i^4 = i^2 \cdot i^2 = +1, \qquad i^5 = i^4 \cdot i = +i, \text{ etc.}$$

Beginning with the fifth power, all the results repeat themselves in the same order. There are only four different values, namely: $+i$, -1, $-i$, $+1$.

57. *If* $x+iy=0$, *then must* $x=0$, $y=0$. Otherwise we should have $x=-iy$; but x is real by hypothesis, and hence x cannot equal $-iy$, which is imaginary.

58. *If* $x+iy=a+ib$, *then* $x=a$, $y=b$. Otherwise we should have $x-a=i(b-y)$, which cannot be, since $x-a$ is real.

59. *The algebraic sum of any number of complex quantities is a complex quantity.*

Suppose we have, say, three complex numbers, x_1+y_1i, x_2+y_2i, x_3+y_3i, then $(x_1+y_1i)+(x_2+y_2i)-(x_3+y_3i)=(x_1+x_2-x_3)+(y_1+y_2-y_3)i$, by the laws of algebra already established. But $x_1+x_2-x_3$ and $y_1+y_2-y_3$ are real, since x_1, x_2, x_3, y_1, y_2, y_3 are real. Hence $(x_1+x_2-x_3)+(y_1+y_2-y_3)i$ is a complex number. The conclusion obviously holds, however many terms there may be in the algebraic sum. For special case where the sum is *real* see Art. 64.

60. *The product of any number of complex numbers is a complex number.*

Consider the product of two complex numbers, x_1+y_1i and x_2+y_2i. We have

$$(x_1+y_1i)(x_2+y_2i)=x_1x_2+y_1y_2i^2+x_1y_2i+x_2y_1i.$$

Hence, bearing in mind the definition of i, we have

$$(x_1+y_1i)(x_2+y_2i)=(x_1x_2-y_1y_2)+(x_1y_2+x_2y_1)i,$$

which proves that the product of two complex numbers is a complex number. The proposition is easily extended to a product of three or more complex numbers. For special case where the product is real see Art. 64.

61. *The quotient of two complex numbers is a complex number.*

We have

$$\frac{x_1+y_1i}{x_2+y_2i}=\frac{(x_1+y_1i)(x_2-y_2i)}{x_2^2-(y_2i)^2}$$

$$=\frac{(x_1x_2+y_1y_2)-(x_1y_2-x_2y_1)i}{x_2^2+y_2^2}$$

$$=\frac{x_1x_2+y_1y_2}{x_2^2+y_2^2}-\left(\frac{x_1y_2-x_2y_1}{x_2^2+y_2^2}\right)i,$$

which proves the proposition.

Cor. I. Since every rational function involves only the operations of addition, subtraction, multiplication, and division, it follows from the above theorems that *every rational function of two or more complex numbers can be reduced to a complex number.*

Cor. II. *If $f(x+yi)$ be any integral function of $x+yi$, having all its coefficients real, and if*

$$f(x+yi)=P+Qi,$$

then $$f(x-yi)=P-Qi,$$

where P and Q are real.

For it is obvious that P can contain only even powers of y, and Q only odd powers of y. If, therefore, we change the sign of y, P will remain unaltered, and Q will simply change its sign. Hence the theorem.

Cor. III. *If $\phi(x+yi)$ be any rational function of $x+yi$, having all its coefficients real, and if*

$$\phi(x+yi)=X+Yi,$$

then $$\phi(x-yi)=X-Yi.$$

EXAMPLES.

1. $3(3+2i)-2(2-3i)+(6+8i)=11+20i.$

2. $(2+3i)(2-3i)(3-5i)=(4+9)(3-5i)=39-65i.$

3. $\dfrac{3+5i}{2-3i}=\dfrac{(3+5i)(2+3i)}{4+9}=-\dfrac{9}{13}+\dfrac{19}{13}i.$

4. $(x+yi)^4=(x^4-6x^2y^2+y^4)+(4x^3y-4xy^3)i.$

62. Two complex numbers which differ only in the sign of their imaginary part are said to be *conjugate*.

Thus $-3-2i$ and $-3+2i$; $-4i$ and $+4i$; $x+yi$ and $x-yi$, are conjugate.

The student has met with conjugate imaginaries in the solution of quadratic equations, where if one root is imaginary, the other is also imaginary, and is conjugate to the first.

63. *If $a+ib$ is a root of an algebraic equation, then also is $a-ib$ a root of the same equation.*

For, let $f(x)=0$ be the equation. If $a+ib$ is a root, we must have $f(a+ib)=0$. This may be written

$$f(a+ib)=0=P+iQ=0;$$

and this requires $P=0$, $Q=0$ (Art. 57). Hence $P-iQ=0$, and $f(a-ib)=P-iQ=0$; hence $a-ib$ is a root of $f(x)=0$.

64. The sum of the conjugate imaginaries, $x+iy$, $x-iy$, is the real quantity $2x$; their difference is the pure imaginary $2iy$.

Their product x^2+y^2 is called the *norm* of either of them.

$$\text{norm}\ (x+iy)=\text{norm}\ (x-iy)=x^2+y^2.$$

The *modulus* of a complex quantity is the positive square root of the norm. Thus, employing the usual symbol,

$$\text{mod}\,(x+iy)=\sqrt{x^2+y^2},$$

$$\text{mod}\,(x-iy)=\sqrt{x^2+y^2}.$$

Rem. When $y=0$, that is, if the complex number be wholly real, then the modulus reduces to $+\sqrt{x^2}$, or x, that is simply the numerical value of x. For example,

$$\text{mod}\,(-3)=+\sqrt{(-3)^2}=+3,\ \text{mod}\,(+5)=+5.$$

EXAMPLES.

$$\text{norm}\,(-3+4i)=(-3)^2+(4)^2=25.$$

$$\text{norm}\,(4-5i)=41.$$

$$\text{mod}\,(-3+4i)=5.$$

$$\text{mod}\,(2-5i)=\sqrt{29}.$$

$$\text{mod}\,(1+\ i)=\sqrt{2}.$$

$$\text{mod}\,(6+8i)=10.$$

65. *If a complex number vanish, its modulus vanishes; and conversely, if the modulus vanish, the complex number vanishes.*

For, if $\quad x+yi=0$, then $x=0$, and $y=0$.

Hence $\quad \sqrt{x^2+y^2}=0.$

Again, if $\sqrt{x^2+y^2}=0$, then $x^2+y^2=0$, hence, since x and y are real, $x=0$ and $y=0$.

66. *If two complex numbers are equal, their moduli are equal.*

For, if $\quad x+yi=x'+y'i$, then $x=x'$, $y=y'$;

hence $\quad \sqrt{x^2+y^2}=\sqrt{x'^2+y'^2}.$

The converse is obviously not true.

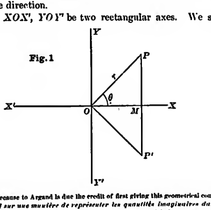
67. Graphic Representation. — Argand's Diagram.* We shall consider now the graphic method of representing complex numbers originally suggested by Argand.

We have seen that the usual representation of positive or negative quantities is by means of distances measured along a straight line, positive quantities being represented by distances measured to the right, negative quantities by distances to the left. For some reasons it is best to say that positive quantities are represented by distances measured to the right, and that the effect of multiplying any quantity by -1 is to reverse the direction; that is, if the quantity is multiplied twice by i, the direction is reversed.

If now the factor i^2, or $i \cdot i$, changes the direction by 180°, then it seems natural to consider i a factor that changes the direction by 90°. It is customary to say that the effect of multiplying by i is to turn the line through an angle of 90° in the positive direction (counter-clockwise). It is evident that the repetition of the operation of using i once as a factor, reverses the direction.

Now, let XOX', YOY' be two rectangular axes. We shall

Fig. 1



* So called because to Argand is due the credit of first giving this geometrical construction in his *Essai sur une manière de représenter les quantités imaginaires dans les constructions géométriques* (1806). See Chrystal's *Algebra*, Vol. I. p. 249. See also Appendix B.

call XOX' the axis of real quantity, YOY' the axis of purely imaginary quantity.

To represent the complex number $x+iy$, we lay off on the x-axis the distance $OM = x$, and on MP, perpendicular to the x-axis, the distance $MP = y$.

Thus the point P is definitely located by the quantity $x+iy$. The distance $OP = r = \sqrt{x^2+y^2} = \text{mod}\,(x+iy)$, and we have $\cos MOP = \cos\theta = \frac{x}{r}$, $\sin\theta = \frac{y}{r}$.

Hence the expression $x+iy$ may be written in the form

$$r(\cos\theta + i\sin\theta).$$

The quantity r is called the *modulus*,* and the angle θ the *argument* of the complex number $x+iy$.

The modulus and argument of $x+iy$ are for brevity represented by the notation

$$\text{mod}\,(x+iy),\ \arg\,(x+iy).$$

EXAMPLE: To write $3+4i$ in the trigonometric form $r(\cos\theta + i\sin\theta)$, we have

$$r = \sqrt{3^2+4^2} = 5,\quad \cos\theta = \tfrac{3}{5},\quad \sin\theta = \tfrac{4}{5},$$

and $$\therefore\ 3+4i = 5(\tfrac{3}{5} + i\tfrac{4}{5}).$$

COR. Of course $x - iy$, or $r(\cos\theta - i\sin\theta)$, represents the point P', the y in this case being measured downward because it is negative. If the argument of $x+iy$ is θ, the argument of $x-iy$ is $2\pi - \theta$, or we may say that two conjugate numbers have the same projection on the x-axis.

68. The Exponential Form of $x+iy$. The following developments for $\cos\theta$, $\sin\theta$, and e^x, which are deduced in works on trigonometry and elementary calculus, are supposed to be known:

* German writers use "absolute value" instead of "modulus," and denote it by the symbol $|x+iy|$. Thus $\sqrt{x^2+y^2} = |x+iy| =$ absolute value of the complex number $x+iy$. So also θ is often called the "amplitude" instead of the "argument."

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots,$$

$$\cos\theta = 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \cdots,$$

$$\sin\theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} + \cdots.$$

From the last two we have

$$\cos\theta + i\sin\theta = 1 + i\theta - \frac{\theta^2}{2!} - i\frac{\theta^3}{3!} + \frac{\theta^4}{4!} + i\frac{\theta^5}{5!} - \cdots.$$

If we define a function $e^{i\theta}$ by the series

$$e^{i\theta} = 1 + i\theta + \frac{(i\theta)^2}{2!} + \frac{(i\theta)^3}{3!} + \frac{(i\theta)^4}{4!} + \frac{(i\theta)^5}{5!} + \cdots$$
$$= 1 + i\theta - \frac{\theta^2}{2!} - i\frac{\theta^3}{3!} + \frac{\theta^4}{4!} + i\frac{\theta^5}{5!} + \cdots,$$

which is entirely analogous to the form for e^x, where x is real, then we have

$$e^{i\theta} = \cos\theta + i\sin\theta,$$

and, consequently,

$$x + iy = r(\cos\theta + i\sin\theta) = re^{i\theta}.$$

Similarly, $\quad x - iy = r(\cos\theta - i\sin\theta) = re^{-i\theta}.$

Cor. I. The following formulæ are sometimes useful:

$$e^{i\pi} = \cos\pi + i\sin\pi = -1,$$

$$e^{-i\pi} = \cos\pi - i\sin\pi = -1,$$

$$e^{i\frac{\pi}{2}} = \cos\frac{\pi}{2} + i\sin\frac{\pi}{2} = i,$$

$$e^{-i\frac{\pi}{2}} = \cos\frac{\pi}{2} - i\sin\frac{\pi}{2} = -i.$$

Cor. II. If $\theta = \frac{\pi}{2}$, and $r = 1$, then $x + iy$ becomes

$$1 \cdot \left(\cos\frac{\pi}{2} + i \sin\frac{\pi}{2}\right) = 1 \cdot i = e^{+i\frac{\pi}{2}}.$$

Hence $e^{+i\frac{\pi}{2}} = i$, is the operator which turns the direction through 90°.

69. Expressing $e^{i\theta}, e^{i\alpha}$ in their respective trigonometric forms, and performing the operations of multiplication and division, we can readily prove the relations:

$$e^{i\theta} \cdot e^{i\alpha} = e^{i(\alpha+\theta)},$$

$$\frac{e^{i\theta}}{e^{i\alpha}} = e^{i(\theta-\alpha)}.$$

Hence, the function $e^{i\theta}$, defined in the last paragraph, obeys the same laws of multiplication and division as the function e^x, where x is real.

70. De Moivre's Theorem. *First*, for n a positive whole number.

If in the equation

$$(x + iy)(a + ib) = mre^{i(\theta+\alpha)},\text{*}$$

we let $a + ib = x + iy$, it becomes

$$(x + iy)^2 = r^2 e^{i \cdot 2\theta} = r^2(\cos 2\theta + i \sin 2\theta);$$

similarly, $(x + iy)^n = r^n \cdot e^{i \cdot n\theta} = r^n(\cos n\theta + i \sin n\theta)$.

Hence, for n a positive whole number,

$$(\cos\theta + i \sin\theta)^n = \cos n\theta + i \sin n\theta.$$

Second, for n a negative whole number.

We know that

* Here $a + ib = me^{i\alpha}$.

$$\frac{x+iy}{a+ib}=\frac{r}{m}e^{i(\theta-a)}.$$

If in this we make

$$(a+ib)=(x+iy)^{n+1}=r^{n+1}e^{i(n+1)\theta}$$

we shall have

$$\frac{x+iy}{(x+iy)^{n+1}}=(x+iy)^{-n}=\frac{r}{r^{n+1}}e^{i[\theta-(n+1)\theta]}=r^{-n}e^{-in\theta}.$$

Hence

$$[r(\cos\theta+i\sin\theta)]^{-n}=r^{-n}[\cos(-n\theta)+i\sin(-n\theta)].$$

$$\therefore\ (\cos\theta+i\sin\theta)^{-n}=\cos n\theta-i\sin n\theta.$$

Hence $(e^{i\theta})^n=e^{in\theta}$, where n is any positive or negative whole number.

Third, n any number.

Suppose that $\theta=\frac{\phi}{t}$, then $e^{i\theta}=e^{i\frac{\phi}{t}}$, and $(e^{i\theta})^t=(e^{i\frac{\phi}{t}})^t=e^{it\theta}=e^{i\phi}$. That is, the tth power of $e^{i\frac{\phi}{t}}$ is $e^{i\phi}$; conversely, one of the tth roots of $e^{i\phi}$ must be $e^{i\frac{\phi}{t}}$;

hence $$(\cos\theta+i\sin\theta)^{\frac{1}{t}}=\cos\frac{\theta}{t}+i\sin\frac{\theta}{t}.$$

Finally, if s and t are any whole numbers, we have

$$(e^{i\theta})^{\frac{s}{t}}=e^{i\frac{s}{t}\theta}=\cos\frac{s}{t}\theta+i\sin\frac{s}{t}\theta;$$

but as s and t are any numbers whatever, $\frac{s}{t}$ may represent any rational or ~~irrational number~~; hence, when n is any number whatever, integer, fractional, or ~~irrational~~, we have

$$(e^{i\theta})^n=e^{in\theta},$$

or $$\therefore\ (\cos\theta+i\sin\theta)^n=\cos n\theta+i\sin n\theta,$$

which is *De Moivre's Theorem*.*

* Abraham de Moivre (1667–1754). The discovery of this theorem by De Moivre revolutionized analytical trigonometry.

71. The Values of $(e^{i\theta})^{\frac{1}{n}}$, for integer value of n.

By definition, we have

$$e^{2i\pi} = \cos 2\pi + i \sin 2\pi = 1;$$

hence $\qquad e^{i\theta} \cdot e^{2i\pi} = e^{i\theta} = e^{i(\theta+2\pi)},$

or, more generally,

$e^{i\theta} = e^{i(\theta+2k\pi)}$, where k is any whole number whatever.

Hence $\qquad (e^{i\theta})^{\frac{1}{n}} = (e^{i(\theta+2k\pi)})^{\frac{1}{n}} = e^{i\frac{\theta+2k\pi}{n}}.$

whence $\qquad (e^{i\theta})^{\frac{1}{n}} = \cos\dfrac{\theta+2k\pi}{n} + i\sin\dfrac{\theta+2k\pi}{n},$

where k may be any whole number. While from this equation the number of values of $(e^{i\theta})^{\frac{1}{n}}$ is apparently infinite, there are really only n different values, for when k has run through the numbers $0, 1, 2, 3, \cdots, n-1$, the values of $(e^{i\theta})^{\frac{1}{n}}$ begin to repeat themselves, as may be readily shown.

72. Solution of the Equation $x^n - 1 = 0$.

This is a special form of the *binomial equation*, the general form of such equations being $x^n = a + b\sqrt{-1}$, where a and b are real quantities. To find the roots of

$$x^n = 1, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

we have $\qquad x^n = e^{2i\pi} = e^{i(2\pi+2k\pi)}.$

Hence $\quad x = e^{i\frac{2\pi+2k\pi}{n}} = \cos\dfrac{2\pi+2k\pi}{n} + i\sin\dfrac{2\pi+2k\pi}{n} \quad . \quad (2)$

For $k = n-1$, we have

$$x = \cos\frac{2n\pi}{n} + i\sin\frac{2n\pi}{n} = \cos 2\pi + i\sin 2\pi = 1.$$

Therefore $+1$ is a root of the equation, $x^n = 1$.

If n is even, we may make $k=\frac{n}{2}-1$, then we have

$$x=\cos\frac{n\pi}{n}+i\sin\frac{n\pi}{n}=\cos\pi+i\sin\pi=-1.$$

Hence, if n is even, both $+1$ and -1 are roots of $x^n=1$. But if n is odd, $+1$ is the only real root. This is evident from the fact that for all values of k, other than $\frac{n}{2}-1$ for n even, and $n-1$ for n even or odd, $\sin\frac{2\pi+2k\pi}{n}$ is not zero, and therefore the root is imaginary.

73. Solution of the Equation $x^n+1=0$. To find the roots of

$$x^n=-1\cdots,$$

we have $\quad x^n=e^{i(\pi+2k\pi)}$, since $e^{i\pi}=-1$.

Hence $\quad x=e^{i\frac{\pi+2k\pi}{n}}=\cos\frac{\pi+2k\pi}{n}+i\sin\frac{\pi+2k\pi}{n}.$

If n is even, the roots are all imaginary, since no even power of a real quantity can be negative; but if n is odd, we may make $k=\frac{n-1}{2}$; then we find $x=\cos\pi+i\sin\pi=-1$. We conclude that when n is odd, there is one and only one real root, -1.

EXAMPLES.

1. Find the cube roots of $+1$.

Here $x^3=1$, and in equation (2), Art. 72, k may be made equal successively to 0, 1, 2, while $n=3$. We thus get for the roots

$$x=\cos\tfrac{2}{3}\pi+i\sin\tfrac{2}{3}\pi=-\tfrac{1}{2}+\tfrac{1}{2}\sqrt{3}\,i,$$

$$x=\cos\tfrac{4}{3}\pi+i\sin\tfrac{4}{3}\pi=-\tfrac{1}{2}-\tfrac{1}{2}\sqrt{3}\,i,$$

$$x=\cos2\pi+i\sin2\pi=+1.$$

2. Solve the equations $x^4=1$, and $x^4=-1$.

3. Solve the equation $x^5 = 1$.

The roots are:

$$\tfrac{1}{4}(\sqrt{5}-1)+\tfrac{1}{4}\sqrt{(10+2\sqrt{5})}\,i,$$
$$-\tfrac{1}{4}(\sqrt{5}+1)+\tfrac{1}{4}\sqrt{(10-2\sqrt{5})}\,i,$$
$$-\tfrac{1}{4}(\sqrt{5}+1)-\tfrac{1}{4}\sqrt{(10-2\sqrt{5})}\,i,$$
$$\tfrac{1}{4}(\sqrt{5}-1)-\tfrac{1}{4}\sqrt{(10+2\sqrt{5})}\,i.$$

4. Solve the equation $x^{10} = 1$.

74. Complex Numbers. — Addition. Let rectangular axes be taken and a point P representing $a+ib$; that is, Art. 67, $OM = a$, $PM = b$, and

$OP = \sqrt{a^2+b^2} = \mu = \text{mod}\,(a+ib)$, and $MOP = \alpha = \arg(a+ib)$.

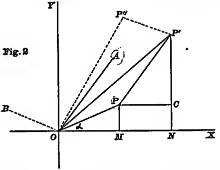


Fig. 2

Let a second complex number $a'+ib'$ be represented by the point A, so that

$$OA = \text{mod}\,(a'+ib'),\quad XOA = \arg(a'+ib').$$

Now the sum of these two complex numbers is

$$a+ib+a'+ib',$$

which may be written in the form

$$a + a' + i(b + b'),$$

and we observe that this sum is represented by the point whose coördinates are $a + a'$, $b + b'$.

To find this point draw PP'' parallel and equal to OA; since PC, $P''C$ are equal to a', b', P'' is the required point, and we have

$$OP'' = \text{mod}\,\{a + a' + i(b + b')\},\quad XOP'' = \arg\,\{a + a' + i(b + b')\}.$$

Therefore, to add two complex numbers, represented by the points A and P, we draw PP'' equal and parallel to OA; then P'' represents the sum of the two complex numbers.

Since OP'' is not greater than $OP + PP''$, it follows that *the modulus of the sum of two complex numbers is less than (or at most equal to) the sum of their moduli.*

To add a third complex number $a'' + ib''$, represented by B, we draw $P''P'''$ parallel and equal to OB. Then P''' represents

$$a + a' + a'' + i(b + b' + b''),$$

which is the sum of the three given complex numbers.

As this mode of representation may be extended to the addition of any number of such quantities, it is evident that, in general, *the modulus of the sum of any number of complex quantities is less than (or at most equal to) the sum of their moduli.*

75. Subtraction. Subtraction can be represented in a similar way. Since P'' represents the sum of P and A, P will represent the difference of P'' and A. To subtract two complex numbers, therefore, we draw from the point representing the minuend a line parallel and equal to the line from the origin to the point representing the subtrahend, but in the opposite direction. We join O to the extremity of this line to find the

modulus of the point which represents the difference of the two given complex numbers.

76. Multiplication and Division. The theorems of Arts. 60 and 61 may readily be proved by De Moivre's Theorem, as follows:

To multiply the two complex numbers $a + ib$, $a' + ib'$, we write them in the form

$$(a + ib) \equiv \rho(\cos\alpha + i\sin\alpha),\quad a' + ib' \equiv \rho'(\cos\alpha' + i\sin\alpha').$$

Then

$$(a + ib)(a' + ib') \equiv \rho\rho'\{\cos(\alpha + \alpha') + i\sin(\alpha + \alpha')\},$$

which proves that *the product of two complex numbers is a complex number, whose modulus is the product of the two moduli, and whose argument is the sum of the two arguments.*

Similarly, we may prove that the product of any number of complex quantities is a complex quantity whose modulus is the product of all the moduli, and whose argument is the sum of all the arguments.

To divide $a + ib$ by $a' + ib'$, we have similarly

$$\frac{a + ib}{a' + ib'} \equiv \frac{\rho}{\rho'}\{\cos(\alpha - \alpha') + i\sin(\alpha - \alpha')\},$$

which proves that *the quotient of two complex numbers is a complex number whose modulus is the quotient of the two moduli, and whose argument is the difference of the two arguments.*

Cor. Similar theorems for involution and evolution are derived at once from De Moivre's Theorem.*

* From the formula

$$(x + iy)^n = r^n(\cos n\theta + i\sin n\theta),$$

it is evident that, in involution, θ increases by arithmetical progression, while r increases by geometrical progression.

CHAPTER V.*

PROPERTIES OF POLYNOMIALS.

77. Reduction to the Form $f(x) = 0$. Any rational integral function of x, $f(x)$, may, as we have seen, be put in the form

$$f(x) \equiv a_0x^n + a_1x^{n-1} + a_2x^{n-2} + a_3x^{n-3} + \cdots + a_{n-1}x + a_n.$$

Any equation in x having rational coefficients can be transformed into an equation of the form $f(x) = 0$, as the following example will show.

EXAMPLE. Reduce $\dfrac{x^{\frac{1}{2}} - x^{\frac{2}{3}}}{1+x} = \dfrac{x^{-1} - 3}{x^{\frac{1}{3}} + 2}$ to the form $F(x) = 0$.

Clearing the given equation of fractions, we obtain

$$x^{\frac{5}{6}} - x + 2x^{\frac{1}{2}} - 2x^{\frac{2}{3}} = x^{-1} + 1 - 3 - 3x,$$

or, multiplying by x to free of negative exponents,

$$x^{\frac{11}{6}} - x^2 + 2x^{\frac{3}{2}} - 2x^{\frac{5}{3}} = 1 - 2x - 3x^2 \quad . \quad . \quad . \quad (1)$$

To transform (1) into another equation with integral exponents, put $x = y^6$, 6 being the least common multiple of the denominators of the fractional exponents of x. Thus we get

$$2y^{12} + y^{11} - 2y^{10} + 2y^9 + 2y^6 - 1 = 0 \quad . \quad . \quad . \quad (2)$$

which is the required form, the roots of (1) and (2) holding the relation $x = y^6$.

* In this and subsequent chapters, we shall consider mainly the real values of x, and shall not enter upon the general discussion of the theory of the complex variable.

EXAMPLES.

Reduce the following expressions to the form $f(x) = 0$:

1. $\frac{3}{x^2} + 2x - \frac{2}{5}x^{\frac{1}{3}} - x^2 = 1.$

2. $\frac{x^2 - 1}{1 + x^{\frac{1}{2}}} = 2 + x^{-2}.$

3. $\sqrt{4 - 5x} = 1 - 3x^{\frac{1}{2}}.$

4. $\sqrt{x - 5x^2} = \sqrt{1 - 2x} - x.$

5. $(x^{\frac{3}{2}} - 3x^{\frac{1}{2}})(1 - x) = (x^{-2} + 1)(x^{-\frac{1}{2}} - 2).$

78. We shall now give two theorems concerning the relative importance of the terms of a polynomial when values very great or very small are assigned to x.

Writing the polynomial in the form

$$a_0x^n\left\{1 + \frac{a_1}{a_0}\frac{1}{x} + \frac{a_2}{a_0}\frac{1}{x^2} + \cdots + \frac{a_{n-1}}{a_0}\frac{1}{x^{n-1}} + \frac{a_n}{a_0}\frac{1}{x^n}\right\},$$

it is plain that its value tends to become equal to a_0x^n as x tends toward ∞. The following theorem will determine a quantity such that the substitution of this, or of any greater quantity, for x will have the effect of making the term a_0x^n exceed the sum of all the others. In what follows we suppose a_0 to be positive; and, in general, in the treatment of polynomials and equations the highest term is supposed to be written with the positive sign.

THEOREM. *If in the polynomial*

$$a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \cdots + a_{n-1}x + a_n$$

the value $\frac{a_k}{a_0} + 1$, *or any greater value, be substituted for* x, *where* a_k *is that one of the coefficients* $a_1, a_2, a_3, \cdots a_n$ *whose numerical value is greatest irrespective of sign, the term containing the highest power of* x *will exceed the sum of all the terms which follow.*

The inequality

$$a_0x^n > a_1x^{n-1} + a_2x^{n-2} + \cdots + a_{n-1}x + a_n$$

is satisfied by any value of x which makes

$$a_0x^n > a_k(x^{n-1} + x^{n-2} + \cdots + x + 1),$$

where a_k is the greatest among the coefficients $a_1, a_2, \cdots a_{n-1}, a_n$, without regard to sign. Summing the geometric series within the brackets, we have

$$a_0x^n > a_k\frac{x^n - 1}{x - 1} \text{ or } x^n > \frac{a_k}{a_0(x-1)}(x^n - 1),$$

which is satisfied if $a_0(x - 1)$ be $>$ or $= a_k$; that is,

$$x > \text{ or } = \frac{a_k}{a_0} + 1.$$

This theorem is useful in supplying, when the coefficients of the polynomial are given numbers, a number such that when x receives values nearer to $+\infty$, the polynomial will preserve constantly a positive sign.

If we change the sign of x, the first term will retain its sign if n be even, and will become negative if n be odd; so that the theorem also supplies a negative value of x, such that for any value nearer to $-\infty$, the polynomial will retain constantly a positive sign, if n be even, and a negative sign, if n be odd.

As illustrative of the use of this theorem, consider the polynomial $10x^3 - 17x^2 + x + 6$.

Here, substituting 10 for a_0 and 17 for a_k, the test formula becomes

$$x > \text{ or } = \tfrac{17}{10} + 1,$$

or

$$x > \text{ or } = 2.7,$$

which shows us that the function $10x^3 - 17x^2 + x + 6$ retains positive values for all positive values of x greater than 2.7, and negative values for all values of x nearer to $-\infty$ than 2.7.

79. We next consider a theorem which shall enable us to determine what term controls the sign of a polynomial when the value of x is indefinitely diminished.

THEOREM. *If in the polynomial*

$$a_0x^n + a_1x^{n-1} + \cdots + a_{n-1}x + a_n$$

the value $\dfrac{a_n}{a_n + a_k}$, *or any smaller value, be substituted for* x, *where* a_k *is the greatest coefficient exclusive of* a_n, *the term* a_n *will be numerically greater than the sum of all the others.*

To prove this, let $x = \dfrac{1}{y}$; then by the theorem of Art. 78, a_k being now the greatest among the coefficients $a_0, a_1, \cdots a_{n-1}$, without regard to sign, the value $\dfrac{a_k}{a_n} + 1$, or any greater value of y, will make

$$a_ny^n > a_{n-1}y^{n-1} + a_{n-2}y^{n-2} + \cdots + a_1y + a_0,$$

that is
$$a_n > a_{n-1}\frac{1}{y} + a_{n-2}\frac{1}{y^2} + \cdots + a_0\frac{1}{y^n};$$

hence the value $\dfrac{a_n}{a_n + a_k}$, or any less value of x will make

$$a_n > a_{n-1}x + a_{n-2}x^2 + \cdots + a_0x^n.$$

Cor. I. This proposition may be stated as follows:

Values so small may be assigned to x *as to make the polynomial*

$$a_{n-1}x + a_{n-2}x^2 + \cdots + a_0x^n$$

less than any assigned quantity.

This statement of the theorem follows at once from the above proof, since a_n may be taken to be the assigned quantity.

Cor. II. Another useful statement of the theorem is as ollows:

When the variable x receives a very small value, the sign of the polynomial

$$a_{n-1}x + a_{n-2}x^2 + \cdots + a_0x^n$$

is the same as the sign of its first term $a_{n-1}x$.

This is evident, if we write the expression in the form

$$x\{a_{n-1} + a_{n-2}x + \cdots + a_0x^{n-1}\}.$$

80. Derived Functions. *Change of form of a polynomial corresponding to an increase or diminution of the variable.*

We shall now examine the form assumed by the polynomial when $x + h$ is substituted for x. Here the resulting form will correspond to an increase or diminution of the variable x, according as h is positive or negative.

The polynomial

$$f(x) \equiv a_0x^n + a_1x^{n-1} + a_2x^{n-2} + a_3x^{n-3} + \cdots + a_{n-1}x + a_n \quad . \quad (A)$$

becomes, when x is changed to $x + h$, $f(x + h)$, or

$$a_0(x+h)^n + a_1(x+h)^{n-1} + a_2(x+h)^{n-2} + \cdots + a_{n-1}(x+h) + a_n.$$

Expanding each term of this expression by the binomial theorem, and arranging the result according to ascending powers of h, we have

$$\begin{aligned}
&a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \cdots + a_{n-2}x^2 + a_{n-1}x + a_n \\
&+ h\{na_0x^{n-1} + (n-1)a_1x^{n-2} + (n-2)a_2x^{n-3} + \cdots + 2a_{n-2}x + a_{n-1}\} \\
&+ \frac{h^2}{1 \cdot 2}\{n(n-1)a_0x^{n-2} + (n-1)(n-2)a_1x^{n-3} + \cdots + 2a_{n-2}\} \\
&+ \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \\
&+ \frac{h^n}{1 \cdot 2 \cdot 3 \cdots n}\{n \cdot (n-1) \cdots 2 \cdot 1\}a_0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (B)
\end{aligned}$$

We observe that the part of this expression independent of h is $f(x)$, and that the successive coefficients of the different powers of h are functions of x of degrees diminishing by unity.

We also see that the coefficient of h may be obtained from $f(x)$ by multiplying each term of $f(x)$ by the exponent of x in that term and diminishing the exponent of x by unity, the sign being retained. The sum of all the terms of $f(x)$ treated in this way will constitute a polynomial, one degree lower than $f(x)$.

This polynomial is called the *first derived function* of $f(x)$, and is usually represented by the notation $f'(x)$. The coefficient of $\frac{h^2}{1 \cdot 2}$ is gotten from $f'(x)$ in exactly the same manner as $f'(x)$ is derived from $f(x)$, or by the operation twice performed on $f(x)$. This coefficient, denoted by $f''(x)$, is called the *second derived function*. In a similar way the *third derived function*, $f'''(x)$, is obtained from $f''(x)$, and so on; so that the expression, B, may be written as follows:

$$f(x+h)=f(x)+f'(x)h+\frac{f''(x)}{1 \cdot 2}h^2+\frac{f'''(x)}{1 \cdot 2 \cdot 3}h^3+\cdots+a_0h^n \quad . \quad (C)$$

EXAMPLE.

Find the result of substituting $x+h$ for x in the polynomial $5x^3-6x^2+8x+4$.

Here
$$f(x)=5x^3-6x^2+8x+4,$$
$$f'(x)=15x^2-12x+8,$$
$$f''(x)=30x-12,$$
$$f'''(x)=30,$$
and the result is
$$5x^3-6x^2+8x+4+(15x^2-12x+8)h+(30x-12)\frac{h^2}{1 \cdot 2}+30 \cdot \frac{h^3}{1 \cdot 2 \cdot 3}.$$

81. Continuity of a Rational Integral Function. THEOREM. *If in a rational and integral function $f(x)$ the value of x be made to vary, by indefinitely small increments, from any quantity a to a greater quantity b, then will $f(x)$ at the same time vary also by indefinitely small increments; that is, $f(x)$ varies continuously with x.*

Suppose x to increase from a to $a+h$. The corresponding increment of $f(x)$ is

$$f(a+h)-f(a),$$

and, by Art. 80, this is equal to

$$f'(a)\,h+f''(a)\frac{h^2}{1\cdot 2}+\cdots+a_0h^n,$$

in which expression all the coefficients $f'(a)$, $f''(a)$, etc., are finite quantities.

Now, by Art. 79, Cor. I, this latter expression may, by taking h small enough, be made to assume a value less than any assigned quantity; so that the difference between $f(a+h)$ and $f(a)$ may be made as small as we please, and will ultimately vanish with h. The same is evidently true during all stages of the variation of x from a to b; thus the theorem is proved.

We should observe that it is not here proved that $f(x)$ *increases* continuously from $f(a)$ to $f(b)$, but simply that it varies *continuously*, for it may sometimes increase and at other times decrease.

82. *Form of the quotient and remainder when a polynomial is divided by a binomial.*

Divide $\quad a_0x^n+a_1x^{n-1}+a_2x^{n-2}+\cdots+a_{n-1}x+a_n$

by $x-h$, and let the quotient be

$$b_0x^{n-1}+b_1x^{n-2}+b_2x^{n-3}+\cdots+b_{n-2}x+b_{n-1}.$$

This we shall represent by Q, and the remainder by R. We have then

$$f(x)\equiv(x-h)\,Q+R.$$

The meaning of this equation is, that when Q is multiplied by $x-h$, and R added, the result must be *identical*, term for term, with $f(x)$.

The right hand side of the identity is

$$\begin{array}{l|l|l|l} b_0x^n + b_1 & x^{n-1} + b_2 & x^{n-2} + \cdots + b_{n-1} & x + R \\ -hb_0 & -hb_1 & -hb_{n-2} & -hb_{n-1} \end{array}$$

Equating the coefficients of x on both sides, we get the following series of equations to determine $b_0, b_1, b_2, \cdots b_{n-1}, R$:

$$\begin{aligned} b_0 &= a_0, \\ b_1 &= b_0h + a_1, \\ b_2 &= b_1h + a_2, \\ b_3 &= b_2h + a_3, \\ &\cdots\cdots \\ b_{n-1} &= b_{n-2}h + a_{n-1}, \\ R &= b_{n-1}h + a_n. \end{aligned}$$

These equations supply a ready method of calculating in succession the coefficients b_0, b_1, b_2 etc., of the quotient, and the remainder R. For this purpose we write the series of operations in the following manner:

$$\begin{array}{cccccc} a_0 & a_1, & a_2, & a_3, & \cdots a_{n-1}, & a_n, \\ & b_0h, & b_1h, & b_2h, & \cdots b_{n-2}h & b_{n-1}h, \\ \hline & b_1 & b_2 & b_3 & b_{n-1}, & R \end{array}$$

In the first line are written down the successive coefficients of $f(x)$. The first term in the second line is obtained by multiplying a_0 (or b_0, which is equal to it) by h. The product b_0h is placed under a_1, and then added to it in order to obtain the term b_1 in the third line. This term, thus obtained, is multiplied in its turn by h, and placed under a_2. The product is added to a_2 to obtain the second figure b_2 in the third line. The repetition of this process furnishes in succession all the coefficients of the quotient, the last figure thus obtained being the remainder. This process, called Horner's *Method of Synthetic Division*, will be made plain by a few examples.

The theorem of this article is known as the "Remainder Theorem."

EXAMPLES.

1. Find the quotient and remainder when

$$2x^4 + 4x^3 - x^2 - 16x - 12 \text{ is divided by } x + 4.$$

Write the coefficients with -4 at their right and proceed as below

$$\begin{array}{rrrrr|l} 2 & 4 & -1 & -16 & -12 & \underline{-4} \\ & -8 & 16 & -60 & 304 & \\ \hline 2 & -4 & +15 & -76 & +292 & \end{array}$$

Thus the quotient is $2x^3 - 4x^2 + 15x - 76$, and the remainder is 292.

2. Find Q and R when $3x^4 - 27x^2 + 14x + 120$ is divided by $x - 6$.

When any term in a polynomial is absent, care must be taken to supply the place of its coefficient by zero in writing down the coefficients of $f(x)$. In this example, therefore, the calculation is as follows:

$$\begin{array}{rrrrr|l} 3 & 0 & -27 & 14 & 120 & \underline{6} \\ & 18 & 108 & 486 & 3000 & \\ \hline 3 & +18 & +81 & +500 & +3120 & \end{array}$$

Hence $Q = 3x^3 + 18x^2 + 81x + 500$, and $R = 3120$.

3. Divide $x^4 - 4x^3 - 8x + 32$ by $x - 4$.

$$\begin{array}{rrrrr|l} 1 & -4 & 0 & -8 & 32 & \underline{4} \\ & 4 & 0 & 0 & -32 & \\ \hline 1 & 0 & 0 & -8 & 0 & \end{array}$$

In this case, therefore, $Q = x^3 - 8$ and $R = 0$, or the division is exact, and 4 is a root of the equation $f(x) = 0$.

4. Find Q and R, when $x^5 - 4x^4 + 7x^3 - 11x - 13$ is divided by $x - 5$.

Ans. $Q = x^4 + x^3 + 12x^2 + 60x + 289$; $R = 1432$.

5. Find Q and R when $x^9 + 3x^7 - 15x^3 + 2$ is divided by $x - 2$.

6. Find Q and R when $x^3 + x^2 - 10x + 113$ is divided by $x + 4$.

83. Tabulation of Functions. Horner's synthetic method of division affords a convenient practical method of calculating the numerical value of a polynomial, with numerical coefficients, when any number is substituted for x.

For, since

$$f(x) \equiv (x - h)\,Q + R$$

is an identical equation, it is satisfied by any value whatever of x.

Let $x = h$, then $f(h) = R$, $x - h$ being equal to zero, and Q remaining finite. Hence the result of substituting h for x in $f(x)$ is the remainder when $f(x)$ is divided by $x - h$, and can be calculated rapidly by the method of the preceding article.

For example, the result of substituting -4 for x in the polynomial of Ex. 1, Art. 82, viz.,

$$2x^4 + 4x^3 - x^2 - 16x - 12,$$

is 292, this being the remainder after division by $x + 4$. This can be verified by actual substitution.

Again, the result of substituting 5 for x in

$$x^5 - 4x^4 + 7x^2 - 11x - 13$$

is 1432, as appears from Ex. 4, Art. 82.

We saw in Art. 81 that as x receives a continuous series of values increasing from $-\infty$ to $+\infty$, $f(x)$ will pass through a corresponding continuous series. If we substitute in succession for x, in a polynomial whose coefficients are given numbers, a series of numbers such as

$$\cdots -5, -4, -3, -2, -1, 0, 1, 2, 3, 4, 5, \cdots,$$

and calculate the corresponding values of $f(x)$, the process may be called the *tabulation of the function.*

EXAMPLES.

1. Tabulate the trinomial $2x^2+x-6$ for the following values of x:

$$-4, -3, -2, -1, 0, 1, 2, 3, 4.$$

Values of x . .	-4	-3	-2	-1	0	1	2	3	4
Values of $f(x)$.	22	9	0	-5	-6	-3	4	15	30

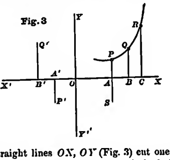
2. Tabulate the polynomial $x^4-4x^3-8x+32$ for the same values of x.

3. Tabulate $x^3-6x^2+11x-6$.

Values of x .	-4	-3	-2	-1	0	1	2	3	4
Values of $f(x)$	-210	-120	-60	-24	-6	0	0	0	$+6$

84. Graphic Representation of a Polynomial. The values of $f(x)$ corresponding to the different real values of x may be conveniently exhibited to the eye by a graphic representation which we shall now explain.

Fig. 3



Let two straight lines OX, OY (Fig. 3) cut one another at right angles, and be produced indefinitely in both directions.

These lines are called the *x-axis* and *y-axis* respectively.

Lines, such as OA, measured on the x-axis, to the right of the y-axis, are regarded as positive; and those, such as OA', measured to the left, as negative. Lines parallel to YY', and above the x-axis, such as AP or $B'Q'$, are positive; and those below XX', such as AS or $A'P'$, are negative. The student of Trigonometry or Analytic Geometry is already acquainted with these conventions.

Any arbitrary length may now be taken on OX as unity, and any number, positive or negative, will be represented by a line measured on XX'. In $f(x)$, give to x the value a and let $OA = a$; calculate $f(a)$; from A draw AP parallel to OY to represent $f(a)$ in magnitude on the same scale as that on which OA represents a, and to represent by its position above or below the line XX' the sign of $f(a)$. $OB = b$, and $BQ = f(b)$, would determine another point Q. Thus, corresponding to the different values of x represented by OA, OB, OC, etc., we shall have a series of points P, Q, R, etc., which, when we suppose the series of values of x indefinitely increased so as to include all numbers between $-\infty$ and $+\infty$, will trace out a continuous curved line. This curve will, by the distances of its several points from the line OX, exhibit to the eye the several values of the function $f(x)$.

The process here explained is also called *tracing the function* $f(x)$, and the curve itself is often called the *graph* of the function.

In the practical application of this method it is well to begin by laying down the points on the curve corresponding to certain small integral values of x, positive and negative. A curve drawn through these points will give at least a general idea of the character of the function. If we wish, at any particular locality, to examine the curve more minutely, we must take several intermediate fractional values of x, and, of course, the closer together such points are taken, the more accurately will the function be delineated.

EXAMPLES.

1. *Trace* the trinomial $-x^2 - 2x + 4$; that is, find its *graph*.

The unit of length taken is one-fourth of the line OE in Fig. 4.

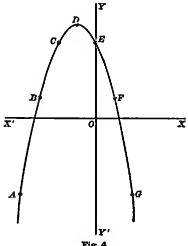


Fig. 4

The values of $f(x)$ corresponding to integral values of x, within the limits of the figure, are as follows:

Values of x,	-4	-3	-2	-1	0	$+1$	$+2$
Values of $f(x)$,	-4	$+1$	$+4$	$+5$	$+4$	$+1$	-4

By means of these values we obtain the positions of seven points on the curve, A, B, C, D, E, F, G. This done, we draw as smooth a curve as we can through these points, which curve is the required *graph*.

2. Trace the polynomial

$$10x^3 - 17x^2 + x + 6.$$

Tabulating the polynomial, we have

Values of x,	-3	-2	-1	0	$+1$	$+2$	$+3$
Values of $f(x)$,	-420	-144	-22	6	0	20	126

We have found, Art. 78, that this function retains positive values for all positive values of x greater than 2.7, and negative values for all values of x nearer to $-\infty$ than -2.7. The graph will, then, if it cuts the axis of x at all, cut it at a point (or points) corresponding to some value (or values) of x between -2.7 and $+2.7$; so, if we wish simply to examine the position of the roots of the equation $f(x) = 0$, the tabulation may be confined to the interval between -2.7 and $+2.7$.

This is a case in which the substitution of integral values only of x gives little help toward the tracing of the curve, and where, consequently, smaller intervals have to be examined. It would be well to tabulate the function for intervals of one-tenth between the integers $-1, 0$; $0, 1$; $1, 2$. This tabulation and the tracing of the curve is left as an exercise for the student.

3. Trace the trinomial $2x^2 + x - 6$.

4. Trace the polynomial $x^4 - 15x^2 + 10x + 24$.

The graph in Ex. 1 cuts the axis of x in two points (a number equal to the degree of the polynomial); in other words, there are two values of x for which the value of the given polynomial is zero; these are the roots of the equation $-x^2 - 2x + 4 = 0$. It will be found that the graph of the polynomial in Ex. 4 cuts the axis of x in four points, corresponding to the roots of the equation

$$x^4 - 15x^2 + 10x + 24 = 0, \text{ viz. } -4, -1, 2, 3.$$

The graph of a given polynomial may not cut the axis of x at all, or may cut it in a number of points less than the degree

of the polynomial. Such cases correspond to the imaginary roots of equations, as will appear more fully in a subsequent chapter. For example, the graph of the polynomial $2x^2+x+2$ will be found to lie entirely above the axis of x. It is evident, by the solution of the equation $2x^2+x+2=0$, that the two values of x which render the polynomial zero are in this case imaginary. Whenever the number of points in which the curve cuts the axis of x falls short of the degree of the polynomial, it is customary to speak of the curve as *cutting the line in imaginary points.*

80, 81

CHAPTER VI.

GENERAL PROPERTIES OF EQUATIONS.

85. We shall first prove some theorems which establish the existence of a real root in an equation in certain cases.

THEOREM. *If two real numbers substituted for x in a rational integral expression $f(x)$ give results with contrary signs, one root at least of the equation $f(x) = 0$ lies between those values of x.*

Let a and b denote the two numbers; then $f(a)$ and $f(b)$ have contrary signs. By Art. 81, as x changes gradually from a to b, the expression $f(x)$ passes without any interruption of value from $f(a)$ to $f(b)$; but since $f(a)$ and $f(b)$ are of contrary signs, the value zero lies between them, so that $f(x)$ must be equal to zero for some value of x between a and b; that is, there is a root of the equation $f(x) = 0$ between a and b.

We do not say that there is *only* one root; and we do not say that if $f(a)$ and $f(b)$ are of the *same* sign there will be no root of the equation $f(x) = 0$ between a and b.

Reference to the graphic method of representation will assist our conception of this theorem, and will enable us to make it more general. It is evident that if there exist two points of the graph of $f(x)$ on *opposite* sides of the axis XX', then the curve between these points must cut that axis an *odd* number of times, and if the two points are on the *same* side of the axis, the curve must cut that axis either not at all or an *even* number of times; thus several values may exist between a and b for which $f(x) = 0$, that is, for which the graph cuts the axis.

For example, in Ex. 2, Art. 84, $x = -1$ gives a negative value (-22), and $x = +2$ gives a positive value (20), and

between these points of the curve there exist *three* points of section with the x-axis, as can be easily shown.

86. THEOREM. *Every equation of an odd degree has at least one real root of a sign opposite to that of its last term.*

This is evident at once from the theorem of the last article. Substitute in succession $-\infty$, 0, ∞ for x in the polynomial $f(x)$. The results are, n being odd (see Art. 78),

for $x=-\infty$, $f(x)$ is negative;

for $x=0$, sign of $f(x)$ is the same as that of a_n;

for $x=+\infty$, $f(x)$ is positive.

If a_n is positive, the equation must have a real root between $-\infty$ and 0, *i.e.* a real negative root; and if a_n is negative, the equation must have a real root between 0 and ∞, *i.e.* a real positive root. The theorem is therefore proved.

87. THEOREM. *Every equation of an even degree, whose last term is negative, has at least two real roots, one positive and the other negative.*

The results of substituting $-\infty$, 0, ∞ are in this case

$$-\infty, +, \qquad 0, -, \qquad +\infty, +;$$

hence there is a real root between $-\infty$ and 0, and another between 0 and $+\infty$; *i.e.* there exist at least one real negative and one real positive root.

88. To prevent mistakes, it is well to call attention to exactly what has been proved in the last two articles.

In Art. 86 it is proved that the equation considered has *at least one* real root: it is not proved that it has only one. In Art. 87 it is proved that the equation considered has *at least two* real roots; it is not proved that it has only two.

89. Existence of a Root. Imaginary Roots. We have now proved the existence of a real root in the case of every equation, except one of an even degree whose last term is positive. Such an equation may have no real root at all. We must then examine whether there may not be cases where the equation has imaginary roots, or whether there may not be in certain cases both real and imaginary values of the variable which satisfy the equation. In Chapter IV we have assumed that such is the case. Let us take a simple example by way of illustration.

In Art. 84 we have seen that the graph of the polynomial

$$f(x) \equiv 2x^2 + x + 2$$

lies entirely above the axis of x, as in Fig. 5. The equation $f(x) = 0$ has no real roots; but it has the two imaginary roots

$$-\frac{1}{4}+\frac{\sqrt{15}}{4}\sqrt{-1},\ -\frac{1}{4}-\frac{\sqrt{15}}{4}\sqrt{-1},$$

as is evident by the solution of the quadratic.

We observe, therefore, though there are no real roots, there are in this case two imaginary expressions which reduce the polynomial to zero.

The corresponding general proposition is that *every rational integral equation has a root, real or imaginary.* Such a root has the general form

$$\alpha + \beta\sqrt{-1},$$

α and β being real finite quantities. This form includes both real and imaginary roots, the former corresponding to the value $\beta = 0$.

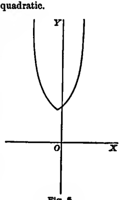


Fig. 5.

The proof of this fundamental theorem, involving principles too intricate to be introduced in an elementary treatise, will not be given, and we shall simply assume the proposition as true, referring the student for the proof to Burnside and Panton's *Theory of Equations*, or Serret's *Cours d'Algèbre Supérieure*, or any advanced work on the subject.*

90. Every Equation of the *n*th Degree has *n* Roots and No More.

It is evident from Art. 83 that if any number h is a root of the equation $f(x)=0$, then $f(x)$ is divisible by $x-h$ without a remainder; for if $f(h)=0$, *i.e.* if h is a root of $f(x)=0$, R must $=0$.

Let the given equation be

$$f(x) \equiv x^n + p_1x^{n-1} + p_2x^{n-2} + \cdots + p_{n-1}x + p_n = 0.$$

This equation must have a root, real or imaginary (Art. 89), which we shall denote by a_1. Let the quotient, when $f(x)$ is divided by $x-a_1$, be $\phi_1(x)$; we have then the identical equation

$$f(x) \equiv (x-a_1)\phi_1(x).$$

Again, the equation $\phi_1(x)=0$, which is of the $(n-1)$th degree, must have a root, which we represent by a_2. Let the quotient obtained by dividing $\phi_1(x)$ by $x-a_2$ be $\phi_2(x)$. Hence

$$\phi_1(x) \equiv (x-a_2)\phi_2(x),$$

and $$\therefore f(x) \equiv (x-a_1)(x-a_2)\phi_2(x),$$

where $\phi_2(x)$ is of the $(n-2)$th degree.

Proceeding in this way, we prove that $f(x)$ consists of the product of n factors, each containing x in the first degree, and a numerical factor $\phi_n(x)$.

If, in the identical equation

$$f(x) \equiv (x-a_1)(x-a_2)\cdots(x-a_n)\phi_n(x),$$

* See also Fine's *Number-System of Algebra*, Arts. 92–94.

we compare the coefficients of x^n, it is plain that $\phi_n(x) = 1$. Thus we prove the identical equation

$$f(x) \equiv (x - \alpha_1)(x - \alpha_2)(x - \alpha_3) \cdots (x - \alpha_{n-1})(x - \alpha_n).$$

It is evident that the substitution of any one of the numbers $\alpha_1, \alpha_2 \cdots \alpha_n$ for x in the right-hand member of this equation will reduce that member to zero, and will, consequently, reduce $f(x)$ to zero; that is, the equation $f(x) = 0$ has for roots the n quantities $\alpha_1, \alpha_2, \alpha_3 \cdots \alpha_{n-1}, \alpha_n$. And it can have no other roots; for if any number other than one of the numbers $\alpha_1, \alpha_2, \alpha_3 \cdots \alpha_n$ be substituted in the right-hand member of the above equation, the factors will all be different from zero, and, therefore, the product cannot vanish.

This theorem, while of no assistance in the solution of the equation $f(x) = 0$, enables us to solve the converse problem; that is, to find the equation whose roots are any n given quantities. The required equation is obtained by multiplying together the n simple factors formed by subtracting from x each of the given roots.

It follows also from the present theorem that, when any (one or more) of the roots of a given equation are known, we can obtain the equation containing the remaining roots by dividing the given equation by the given binomial factor or factors. The quotient will be the required polynomial composed of the remaining factors.

EXAMPLES.

1. Find the equation whose roots are

$$2, \ -1, \ -4, \ +3. \qquad \textit{Ans. } x^4 - 15x^2 + 10x + 24 = 0.$$

2. Two of the roots of the equation

$$x^4 - 5x^3 - 13x^2 + 53x + 60 = 0$$

are $-3, +4$; find the other roots. Use the method of division of Art. 82.

3. Find the equation whose roots are

$$-2, \quad 0, \quad +1, \quad +5.$$

4. In the equation

$$x^3 - 3x^2 - 16x + 48,$$

one root is -4; find the other roots.

5. Solve the equation

$$x^4 - 16x^3 + 86x^2 - 176x + 105 = 0,$$

two roots being 1 and 7. *Ans.* Other roots 3, 5.

6. Form the equation whose roots are

$$-\tfrac{4}{5}, \; 2, \; +\tfrac{1}{3}. \quad \textit{Ans. } 15x^3 + 37x^2 + 12x - 4 = 0.$$

7. Solve the equation

$$x^4 - 4x^3 - 8x + 32 = 0,$$

two roots being $-1 + \sqrt{-3}$, $-1 - \sqrt{-3}$.

8. Solve the cubic equation

$$x^3 - 1 = 0.$$

Here it is evident that $x = 1$ satisfies the equation. Divide by $x - 1$, and solve the resulting quadratic to get the other two roots.

9. Solve the cubic equation

$$x^3 + 1 = 0.$$

91. Equal Roots. It is evident that the n factors of which a polynomial $f(x)$ consists need not be all different from one another. The factor $x - a$, for example, may occur in the second or any higher power not superior to n. In this case two or more of the n roots of $f(x)$ are equal to one another, and the root a is called a multiple root of the equation, — double, triple, etc., according to the number of times the factor is repeated.

Equal roots form the connecting link between real and imaginary roots. A reference to the graphic construction (Art. 84) will make this plain. Or, returning to the equation given in Art. 49, we know that the two roots of the equation $ax^2+bx+c=0$, are *real*, if $b^2>4ac$, *equal*, if $b^2=4ac$, and *imaginary*, if $b^2<4ac$.

92. Theorem. *In an equation with real coefficients, complex roots occur in pairs.*

Let $f(x)$ be a rational, integral function of x in which the coefficients are all real; then if $\alpha+\beta\sqrt{-1}$ is a root of the equation $f(x)=0$, $\alpha-\beta\sqrt{-1}$ will also be a root.

For when $\alpha+\beta\sqrt{-1}$ is put for x, the function $f(x)$ takes the form $P+Q\beta\sqrt{-1}$, where P and Q involve *even* powers of β. Now *as the coefficients in* $f(x)$ *are supposed real*, $\sqrt{-1}$ cannot occur except with some odd power of β. If then $\alpha-\beta\sqrt{-1}$ be substituted for x in $f(x)$, the result will be obtained by changing the sign of β in the result obtained by substituting $\alpha+\beta\sqrt{-1}$ for x; the result is therefore $P-Q\beta\sqrt{-1}$. (Art. 61, Cor. 2.)

Now if $\alpha+\beta\sqrt{-1}$ is a root of $f(x)=0$, then

$$P+Q\beta\sqrt{-1}=0,$$

and, therefore, Art. 57, since β is not zero,

$$P=0, \text{ and } Q=0.$$

Hence $$P-Q\beta\sqrt{-1}=0,$$

and $\alpha-\beta\sqrt{-1}$ is also a root of $f(x)=0$. Thus the total number of imaginary roots in an equation with real coefficients is always even.

Note. A proof exactly similar to that above given shows that *surd roots, of the form* $a\pm\sqrt{\gamma}$, *enter in pairs equations whose coefficients are rational.*

EXAMPLES.

1. Form a rational cubic equation which shall have for two of its roots

$$1, 3-2\sqrt{-1}.$$

2. Form a rational equation which shall have for two of its roots

$$1+5\sqrt{-1},\ 5-\sqrt{-1}.$$

Ans. $x^4-12x^3+72x^2-312x+676=0.$

3. Solve the equation

$$x^4-x^3-8x^2+8=0,$$

which has a root $1+\sqrt{5}$.

4. Solve the equation

$$2x^3-x^2-6x+77,$$

one root being $2+\sqrt{-7}$. *Ans.* $2\pm\sqrt{-7},\ -\frac{7}{2}$.

93. Descartes' Rule of Signs. This celebrated theorem of Descartes* establishes an interesting and useful relation between the number of changes of sign of the first member of an equation, $f(x)=0$, and the number of real roots, and, thereby, enables us to find a superior limit to the number of positive and negative real roots of an equation.

Definition. When each term of a set of terms has one of the signs $+$ or $-$ before it, then in considering the terms in order, a *continuation* is said to occur when a sign is the same as the immediately preceding sign, and a *change*† is said to occur when a sign is contrary to the immediately preceding sign. Thus in the expression

$$x^8-2x^7-3x^6+4x^5+x^4+2x^3-3x^2-x+1$$

* René Descartes (1596–1650).

† Instead of "continuation" and "change" the terms *permanence* and *variation* are often used.

there are four continuations and four changes. It is obvious that in any *complete* equation the number of continuations together with the number of changes is equal to the number which expresses the degree of the equation. If in any complete equation we put $-x$ for x, the continuations and changes in the original equation become respectively changes and continuations in the new equation.

(a) Positive Roots.

THEOREM. *No equation can have more positive real roots than it has changes of sign from* $+$ *to* $-$, *and from* $-$ *to* $+$, *in the terms of its first member.*

Let the signs of a polynomial taken at random succeed each other in the following order:

$$+ + - + - - + + + - + - +$$

In this there are in all eight changes of sign. It is proposed to show that if this polynomial be multiplied by a binomial whose signs, corresponding to a positive root, are $+-$, the resulting polynomial will have at least one more change of sign than the original. Writing down only the signs that occur in the operation, we have

$$\begin{array}{cccccccccccccc}
+ & + & - & + & - & - & + & + & + & - & + & - & + & \\
+ & - & & & & & & & & & & & & \\
\hline
+ & + & - & + & - & - & + & + & + & - & + & - & + & \\
 & - & - & + & - & + & + & - & - & - & + & - & + & - \\
\hline
+ & \pm & - & + & - & \mp & + & \pm & \pm & - & + & - & + & -
\end{array}$$

Here, in the result the ambiguous sign $\pm$ is placed wherever there are two terms with different signs to be added. We readily see that in this case, and in any other arrangement, the effect of the process is to introduce the ambiguous sign wherever the sign $+$ follows $+$, or $-$ follows $-$, in the original polynomial. The number of variations of sign is never diminished, and there is always one variation added at the

end. By trying different arrangements of signs, it is easy to convince ourselves that, in even the most unfavorable case — that, namely, in which the continuations of sign in the original remain continuations in the resulting polynomial, — there is one variation added. We may conclude in general that the effect of the multiplication of a polynomial by a binomial $x - \alpha$ is to introduce at least one change of sign.

Now suppose we have a polynomial formed of the product of the factors corresponding to the negative and imaginary roots of an equation. The effect of multiplying this by each of the factors $x - \alpha$, $x - \beta$, $x - \gamma$, etc., corresponding to the positive roots α, β, γ, etc., is to introduce at least one change of sign for each; so that when the complete product is formed containing all the roots, we conclude that the resulting polynomial has *not more* positive roots than there are changes of sign.

(b) Negative Roots.

THEOREM. *No equation can have a greater number of negative roots than there are changes of sign in the terms of the polynomial* $f(-x)$.

Now, if $-x$ be substituted for x in the equation $f(x) = 0$, the resulting equation will have the same roots as the original, except that their signs will be changed; for, from the identical equation

$$f(x) \equiv (x - \alpha_1)(x - \alpha_2)(x - \alpha_3) \cdots (x - \alpha_n),$$

we derive

$$f(-x) \equiv (-1)^n (x + \alpha_1)(x + \alpha_2)(x + \alpha_3) \cdots (x + \alpha_n).$$

From this it is evident that the roots of $f(-x) = 0$ are

$$-\alpha_1, \ -\alpha_2, \ -\alpha_3, \cdots -\alpha_n.$$

Hence the negative roots of $f(x)$ are positive roots of $f(-x)$, and our theorem for negative roots is true.

EXAMPLES.

1. If the coefficients in $f(x)$ are all positive, the equation $f(x) = 0$ has no positive root.

2. If the coefficients in any complete equation be alternately positive and negative, the equation cannot have a negative root.

3. If an equation consist of a number of terms, whose coefficients are positive followed by a number of terms whose coefficients are negative, it has one positive root and no more.
Apply Art. 85 and Art. 93.

4. If an equation contain only even powers of x, and if all the coefficients have positive signs, it cannot have a real root.

5. If an equation contain only odd powers of x, and if all the coefficients have positive signs, it has the root zero and no other real root.

6. Find an inferior limit to the nnmber of imaginary roots of the eqnation

$$x^6 - 3x^2 - x + 1 = 0.$$

Here, Art. 93, the arrangement of signs for $f(x) = 0$

$$+ \quad - \quad - \quad +$$

exhibits two changes of signs, hence there cannot be more than two positive roots; and, examining the arrangement for $f(-x) = 0$, $+ \; - \; + \; +$, we find again two changes of sign, so there cannot be more than two negative roots. As there are six roots in all, it follows that there must be at least two imaginary roots.

7. Find an inferior limit to the number of imaginary roots of the equation

$$x^6 + 3x^4 + 4x^2 + 2x - 6 = 0.$$

Ans. At least four imaginary roots.

8. Find the nature of the roots of the equation

$$x^4 + 15x^3 + 7x - 11 = 0.$$

Ans. One positive, 1 negative, 2 imaginary.

9. Show that the equation

$$x^3 + qx + r = 0,$$

where q and r are essentially positive, has one negative and two imaginary roots.

10. Find the nature of the roots of the equation

$$x^3 - qx + r = 0.$$

11. Show that the equation

$$x^n - 1 = 0$$

has, when n is even, two real roots, -1 and $+1$, and no other real root; and, when n is odd, the real root 1, and no other real root.

12. Show that the equation

$$x^n + 1 = 0$$

has, when n is even, no real root; and, when n is odd, the real root -1, and no other real root.

CHAPTER VII.

RELATIONS BETWEEN ROOTS AND COEFFICIENTS.—SYMMETRIC FUNCTIONS.

94. *Relations between the roots and coefficients of an equation.* Representing the n roots of the equation

$$x^n + p_1x^{n-1} + p_2x^{n-2} + \cdots + p_{n-1}x + p_n \quad . \quad . \quad . \quad (1)$$

by $\alpha_1, \alpha_2, \alpha_3, \cdots \alpha_n$, we have the identity

$$\begin{aligned} &x^n + p_1x^{n-1} + p_2x^{n-2} + \cdots + p_{n-1}x + p_n \\ &\equiv (x-\alpha_1)(x-\alpha_2)(x-\alpha_3)\cdots(x-\alpha_n) \quad . \quad . \quad . \quad (2) \end{aligned}$$

When the factors of the second member of this identity are multiplied together, the highest power of x in the product is x^n, and the coefficient of this term is unity. The coefficient of the second term, x^{n-1}, is $-\alpha_1-\alpha_2-\alpha_3-\alpha_4\cdots-\alpha_n$; that is, the sum of the roots with their signs changed; the coefficient of x^{n-2} is the sum of the products of the roots taken two and two; the coefficient of x^{n-3} is the sum of the products of the roots taken three at a time, with their signs changed; and so on, the last term being the product of all the roots with their signs changed. Therefore, equating coefficients of like powers of x on each side of the identity (2), we have

$$\left.\begin{aligned} p_1 &= -(\alpha_1+\alpha_2+\alpha_3+\cdots+\alpha_n) \\ p_2 &= (\alpha_1\alpha_2+\alpha_1\alpha_3+\cdots) \\ p_3 &= -(\alpha_1\alpha_2\alpha_3+\alpha_1\alpha_3\alpha_4+\cdots) \\ &\cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \\ p_n &= (-1)^n\alpha_1\alpha_2\alpha_3\cdots\alpha_{n-1}\alpha_n \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (3)$$

These results give us the following relations between the roots and coefficients:

In every algebraic equation, the coefficient of whose highest term is unity, the coefficient p_1 of the second term, with its sign changed, is equal to the sum of the roots.

The coefficient p_2 of the third term is equal to the sum of the products of the roots taken two by two.

The coefficient p_3 of the fourth term, with its sign changed, is equal to the sum of the products of the roots taken three by three, and so on, the signs of the coefficients being alternately negative and positive, till finally that function is reached which consists of the product of the n roots.

When the coefficient a_0 of x^n is not unity (Art. 51), we must divide each term of the equation by it.

Cor. I. Every root of an equation is a divisor, whole or fractional, of the absolute term of the equation.

Cor. II. If the roots of an equation be all positive, the coefficients (including that of the highest power of x) will be alternately positive and negative; and if the roots be all negative, the coefficients will be all positive.

95. It might perhaps be supposed that the relations given in the preceding article would enable us to find by elimination the roots of any proposed equation; for they furnish equations involving the roots, and the number of these equations is the same as the number of the roots. But this is not the case, for, on attempting this elimination, we merely *reproduce the proposed equation itself*, as the following example will show:

Let α, β, γ be the roots of the cubic equation

$$x^3 + p_1x^2 + p_2x + p_3 = 0 \quad . \quad . \quad . \quad . \quad . \quad (1)$$

We have, by Art. 94,

$$p_1 = -(\alpha + \beta + \gamma),$$
$$p_2 = \alpha\beta + \alpha\gamma + \beta\gamma.$$
$$p_3 = -\alpha\beta\gamma.$$

Multiplying the first of these equations by α^2, the second by α, and adding the three, we find

$$p_1\alpha^2 + p_2\alpha + p_3 = -\alpha^3,$$

or

$$\alpha^3 + p_1\alpha^2 + p_2\alpha + p_3 = 0,$$

which is the given cubic with α in the place of x, and, therefore, we are no nearer the solution of (1) than we were at first.

Thus, although the equations (3) afford no aid in the general solution of the equation, they are often useful in facilitating the solution of numerical equations when any particular relations among the roots are known to exist, as will be made apparent by the following examples.

EXAMPLES.

1. Solve the equation

$$x^3 - 3x^2 + 4 = 0,$$

two of its roots being equal.

Let α, α, β be the three roots. We have

$$2\alpha + \beta = 3,$$
$$\alpha^2 + 2\alpha\beta = 0,$$

from which we find $\alpha = 2$, $\beta = -1$. The roots are 2, 2, -1.

2. Solve the equation

$$x^3 - 5x^2 - 16x + 80 = 0,$$

the sum of two of its roots being zero.

Let the roots be α, β, γ. We have then

$$\alpha + \beta + \gamma = 5,$$
$$\alpha\beta + \alpha\gamma + \beta\gamma = -16,$$
$$\alpha\beta\gamma = -80.$$

Taking $\beta + \gamma = 0$, we get $\alpha = 5$, $\beta = 4$, $\gamma = -4$. Thus the three roots are 5, 4, -4.

3. The equation

$$x^4 - 4x^3 - 12x^2 + 32x + 64 = 0$$

has two pairs of equal roots; find them.

4. Solve the equation

$$x^3 - 9x^2 + 14x + 24 = 0,$$

two of whose roots are in the ratio of 3 to 2.

Let the roots be α, β, γ, with the relation $2\alpha = 2\beta$.

Ans. The roots are 6, 4, − 1.

5. Solve the equation

$$x^4 + 2x^3 - 21x^2 - 22x + 40 = 0,$$

whose roots are in arithmetical progression. Assume for the roots $\alpha - 3\delta$, $\alpha - \delta$, $\alpha + \delta$, $\alpha + 3\delta$.

6. Solve the equation

$$8x^4 - 30x^3 + 35x^2 - 15x + 2 = 0,$$

whose roots are in geometrical progression. Assume for the roots $\frac{\alpha}{\rho^3}$, $\frac{\alpha}{\rho}$, $\alpha\rho$, $\alpha\rho^3$. *Ans.* $\frac{1}{4}$, $\frac{1}{2}$, 1, 2.

7. Solve the equation

$$x^3 - 3x^2 - x + 3 = 0,$$

whose roots are in arithmetical progression.

8. Solve the equation

$$24x^3 - 26x^2 + 9x - 1 = 0,$$

whose roots are in harmonic progression. *Ans.* $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$.

9. Solve the equation

$$x^4 + 15x^3 + 70x^2 + 120x + 64 = 0,$$

whose roots are in geometric progression.

10. The equation

$$3x^4 - 25x^3 + 50x^2 - 50x + 12 = 0$$

has two roots whose product is 2; find all the roots.

96. Derived Functions. In order to examine an equation for equal roots, it will be found convenient to express the *derived functions* (Art. 80) in another form.

Let the roots of the equation $f(x)=0$ be $a_1, a_2, a_3, \cdots a_n$. We have

$$f(x) \equiv (x-a_1)(x-a_2)(x-a_3)\cdots(x-a_n).$$

In this identical equation substitute $h+x$ for x:

$$\begin{aligned} f(h+x) &= (h+x-a_1)(h+x-a_2)\cdots(h+x-a_n) \\ &= h^n + q_1 h^{n-1} + q_2 h^{n-2} + \cdots + q_{n-1}h + q_n, \end{aligned}$$

where

$$q_1 = x-a_1+x-a_2+x-a_3+\cdots+x-a_n,$$

$$q_2 = (x-a_1)(x-a_2)+(x-a_1)(x-a_3)+\cdots+(x-a_{n-1})(x-a_n),$$

. . . .

$$\begin{aligned} q_{n-1} = {} & (x-a_2)(x-a_3)\cdots(x-a_n)+(x-a_1)(x-a_3)\cdots(x-a_n)+ \\ & \cdots + (x-a_1)(x-a_2)\cdots(x-a_{n-1}), \end{aligned}$$

$$q_n = (x-a_1)(x-a_2)(x-a_3)\cdots(x-a_n).$$

Also we have, by Art. 80,

$$f(h+x) = f(x) + f'(x)h + \frac{f''(x)}{1\cdot 2}h^2 + \cdots + h^n.$$

Equating the two expressions for $f(h+x)$, we obtain

$$f(x) = (x-a_1)(x-a_2)\cdots(x-a_n),$$

$$f'(x) = (x-a_2)(x-a_3)\cdots(x-a_n)+\cdots, \text{ as above written,}$$

$$\frac{f''(x)}{1\cdot 2} = \text{the similar value of } q_{n-2} \text{ in terms of } x \text{ and the roots,}$$

. . . .

The value of $f'(x)$ may be conveniently written as follows:

$$f'(x) = \frac{f(x)}{x-a_1} + \frac{f(x)}{x-a_2} + \cdots + \frac{f(x)}{x-a_n}.$$

97. Multiple Roots.

THEOREM. *A multiple root of the order p of the equation $f(x)=0$ is a multiple root of the order $p-1$ of the first derived equation $f'(x)=0$.*

This follows at once from the expression given for $f'(x)$ in the preceding article; for, if the factor $(x-a_1)^p$ occurs in $f(x)$, that is, if $a_1=a_2=\cdots=a_p$, we have

$$f'(x)=\frac{pf(x)}{x-a_1}+\frac{f(x)}{x-a_{p+1}}+\cdots+\frac{f(x)}{x-a_n}.$$

Each term of this will still have $(x-a_1)^p$ as a factor, except the first, which will have $(x-a_1)^{p-1}$ as a factor; hence $(x-a_1)^{p-1}$ is a factor in $f'(x)$.

Cor. I. Any root which occurs p times in the equation $f(x)=0$ occurs in degrees of multiplicity diminishing by unity in the first $p-1$ derived equations.

Since $f''(x)$ is derived from $f'(x)$ in the same manner as $f'(x)$ is from $f(x)$, it is evident by the above theorem that $f''(x)$ will contain $(x-a_1)^{p-2}$ as a factor. The next derived function, $f'''(x)$, will contain $(x-a_1)^{p-3}$; and so on.

98. Determination of Multiple Roots.

From the preceding article it is obvious that if $f(x)$ and $f'(x)$ have a common factor $(x-a)^{p-1}$, $(x-a)^p$ will be a factor in $f(x)$; hence a is a root of $f(x)$ of multiplicity p. In the same way, it appears that if $f(x)$ and $f'(x)$ have other common factors $(x-\beta)^{q-1}$, $(x-\gamma)^{r-1}$, $(x-\delta)^{s-1}$, etc., the equation $f(x)=0$ will have q roots equal to β, r roots equal to γ, s roots equal to δ, etc.

Hence, in order to examine an equation $f(x)$ for equal roots and to determine these roots, if such exist, we must find the highest common factor of $f(x)$ and $f'(x)$. Let this be $F_1(x)=0$. The solution of $F_1(x)=0$ will give the equal roots.

EXAMPLES.

1. Find the multiple roots of the equation

$$x^3 - 7x^2 + 16x - 12 = 0.$$

Here the H. C. F. of $f(x) \equiv x^3 - 7x^2 + 16x - 12$, and $f'(x) \equiv 3x^2 - 14x + 16$ is $x - 2$; hence $(x-2)^2$ is a factor in $f(x)$. The other factor is $x - 3$, hence the roots of the equation are 2, 2, 3.

Whenever, after determining the multiple factors of $f(x)$, we wish to get the remaining factors, it will be convenient to apply Horner's method of division (Art. 82). In this example we would divide twice by $x - 2$, the calculation being represented as follows

$$\begin{array}{rrrr} 1 & -7 & +16 & -12 \\ & 2 & -10 & +12 \\ \hline 1 & -5 & 6 & 0 \\ & 2 & -6 & \\ \hline 1 & -3 & 0 & \end{array}$$

Thus 1 and -3 being the two coefficients left, the third factor is $x - 3$. This operation verifies the previous result, the remainders after each division vanishing as they ought.

2. Find the multiple roots, and the remaining factor of the equation

$$x^5 - 10x^2 + 15x - 6 = 0.$$

The H. C. F. of $f(x)$ and $f'(x)$ is $x^2 - 2x + 1$. Hence $(x-1)^3$ is a factor in $f(x)$. Dividing three times in succession by $x - 1$, we obtain

$$f(x) \equiv (x-1)^3(x^2 + 3x + 6).$$

Find the multiple roots of the following equations:

3. $x^3 + x^2 - 16x + 20 = 0.$

4. $x^4 - 2x^3 - 11x^2 + 12x + 36 = 0.$

5. $x^4 - 11x^2 + 18x - 8 = 0.$

6. $x^4 - 11x^3 + 44x^2 - 76x + 48 = 0.$

Ans. $f(x) \equiv (x-2)^2(x-3)(x-4).$

7. $2x^4 - 12x^3 + 19x^2 - 6x + 9 = 0.$

Ans. The roots are $3, 3, +\frac{1}{2}\sqrt{-2}, -\frac{1}{2}\sqrt{-2}.$

8. Show that the binomial equation

$$x^n - p^n = 0$$

cannot have equal roots.

9. Apply the method of Art. 98 to determine the condition that the cubic

$$x^3 + 3Hx + G = 0$$

should have a pair of equal roots. *Ans.* $G^2 + 4H^3 = 0.$

The ordinary process of finding the H. C. F. of $f(x)$ and $f'(x)$ may often become very laborious. It is chiefly in connection with Sturm's theorem (Art. 118) that the operation is of any practical value. Multiple roots of equations of degrees inferior to the sixth can be determined more readily by trial.

99. Theorem. — *In passing continuously from a value $a-h$ of x a little less than a real root a of the equation $f(x) = 0$ to a value $a+h$ a little greater, the polynomials $f(x)$ and $f'(x)$ have unlike signs immediately before the passage through the root, and like signs immediately after.*

Substituting $a-h$ in $f(x)$ and $f'(x)$, and expanding, we have

$$f(a-h) = f(a) - f'(a)h + \frac{f''(a)}{1 \cdot 2}h^2 - \cdots,$$

$$f'(a-h) = \qquad f'(a) \quad - f''(a)h \quad + \cdots.$$

Now, since $f(a) = 0$, the signs of these expressions, depending on those of their first terms, are unlike. When the sign of h is changed, the signs of $f(a+h)$ and $f'(a+h)$ are like. Hence the theorem.

100. The Cube Roots of Unity. Equations of the forms

$$x^n - p = 0, \ x^n + p = 0,$$

are called *binomial equations.* We shall see later that such equations are intimately connected with the more special forms

$$x^n - 1 = 0, \ x^n + 1 = 0,$$

the roots of the first of which are called the *n nth roots of unity.* We shall here consider the simple case of the binomial cubic.

We have seen (Ex. 1, Art. 73) that the roots of the cubic

$$x^3 - 1 = 0$$

are $\quad 1, \ -\frac{1}{2} + \frac{1}{2}\sqrt{-3}, \ -\frac{1}{2} - \frac{1}{2}\sqrt{-3}.$

(See also Ex. 8, Art. 90.)

If either of the imaginary roots be represented by ω, the other is easily seen to be ω^2 by actual squaring. We have then the identity

$$x^3 - 1 \equiv (x-1)(x-\omega)(x-\omega^2).$$

Changing x into $-x$, we get the following identity also:

$$x^3 + 1 \equiv (x+1)(x+\omega)(x+\omega^2),$$

which gives the roots of $x^3 + 1 = 0$.

Whenever ω raised to any higher power than the second presents itself, it can be replaced by ω, or ω^2, or 1; for example

$$\omega^4 = \omega^3 \cdot \omega = \omega, \ \omega^5 = \omega^3 \cdot \omega^2 = \omega^2,$$

$$\omega^6 = \omega^3 \cdot \omega^3 = 1, \text{ etc.}$$

By the first or second of equations (3), of Art. 94, we have the following property:

$$1 + \omega + \omega^2 = 0.$$

Cor. It is important to observe that, corresponding to the n nth roots of unity, there are n nth roots of any quantity.

The roots of the equation

$$x^n - a = 0$$

are the n nth roots of a.

The three cube roots, for example, of a are

$$\sqrt[3]{a},\ \omega\sqrt[3]{a},\ \omega^2\sqrt[3]{a},$$

where $\sqrt[3]{a}$ represents the ordinary (real) cube root. Each of these values satisfies the cubic equation $x^3 - a = 0$.

Thus, besides the ordinary cube root 3, the number 27 has the two imaginary cube roots

$$-\tfrac{3}{2} + \tfrac{3}{2}\sqrt{-3},\ -\tfrac{3}{2} - \tfrac{3}{2}\sqrt{-3},$$

as the student can easily verify by actual cubing.

EXAMPLES.

1. Show that the product

$$(\omega m + \omega^2 n)(\omega^2 m + \omega n)$$

is rational. *Ans.* $m^2 - mn + n^2$.

2. Show that the product

$$(\alpha + \omega\beta + \omega^2\gamma)(\alpha + \omega^2\beta + \omega\gamma)$$

is rational.

3. Form the equation whose roots are $m + n$, $\omega m + \omega^2 n$, $\omega^2 m + \omega n$.

101. Symmetric Functions of the Roots. Symmetric functions of the roots of an equation are those which are not altered if any two of the roots be interchanged. For example, if α, β, γ are the roots of a cubic equation, $\alpha + \beta + \gamma$, $\alpha\beta + \alpha\gamma + \beta\gamma$, $\alpha\beta\gamma$ are symmetric functions, for all the roots are involved alike. The functions p_1, p_2, p_3, etc., of Equation 3,

Art. 94, are the simplest symmetric functions of the roots, each root entering in the first degree only in any one of them. We can often, as shown by some examples appended to this article, obtain the values of a great variety of symmetric functions in terms of the coefficients of the equation whose roots we are considering.

A symmetric function is usually represented by the Greek letter Σ attached to one term of it, from which, by analogy, the entire expression may be written down.

Thus, in the case of a cubic, whose roots are α, β, γ,

$$\Sigma\alpha^2\beta^2 \equiv \alpha^2\beta^2 + \alpha^2\gamma^2 + \beta^2\gamma^2,$$

where all possible products in pairs are taken, and all the terms added after each is separately squared.

Again, $\Sigma\alpha^2\beta \equiv \alpha^2\beta + \alpha^2\gamma + \beta^2\gamma + \beta^2\alpha + \gamma^2\alpha + \gamma^2\beta,$

where all possible permutations of the roots, two by two, are taken, and the first root in each term then squared.

In the case of a biquadratic, we have

$$\Sigma\alpha^2\beta^2 \equiv \alpha^2\beta^2 + \alpha^2\gamma^2 + \alpha^2\delta^2 + \beta^2\gamma^2 + \beta^2\delta^2 + \gamma^2\delta^2.$$

We give a few examples, which may serve to give the student some insight into the formation of this class of functions.

EXAMPLES.

1. Find the value of $\Sigma\alpha^2\beta$ of the roots of the cubic equation

$$x^3 + px^2 + qx + r = 0.$$

Multiplying together the equations

$$\alpha + \beta + \gamma = -p,$$

$$\beta\gamma + \gamma\alpha + \alpha\beta = q,$$

we obtain $$\Sigma\alpha^2\beta + 3\alpha\beta\gamma = -pq;$$

hence $$\Sigma\alpha^2\beta = 3r - pq.$$

2. Find for the same cubic the value of $\alpha^2+\beta^2+\gamma^2$.

Ans. $\Sigma\alpha^2 = p^2 - 2q$.

3. Find for the same cubic the value of

$$\alpha^3+\beta^3+\gamma^3.$$

Multiplying the values of $\Sigma\alpha$ and $\Sigma\alpha^2$, we obtain

$$\alpha^3+\beta^3+\gamma^3+\Sigma\alpha^2\beta = -p^3+2pq;$$

hence, by Ex. 1, $\quad \Sigma\alpha^3 = -p^3+3pq-3r.$

4. Find for the same cubic the value of

$$\Sigma\alpha^2\beta^2.$$

5. If $\alpha, \beta, \gamma, \delta$ are the roots of the biquadratic equation

$$x^4+px^3+qx^2+rx+s=0,$$

find the value of the symmetric function

$$\Sigma\alpha^2\beta\gamma \equiv \alpha^2\beta\gamma+\alpha^2\beta\delta+\alpha^2\gamma\delta+\beta^2\alpha\gamma+\beta^2\alpha\delta+\beta^2\gamma\delta+\gamma^2\alpha\beta+\gamma^2\alpha\delta$$
$$+\gamma^2\beta\delta+\delta^2\alpha\beta+\delta^2\alpha\gamma+\delta^2\beta\gamma.$$

Multiplying together

$$\alpha+\beta+\gamma+\delta=-p,$$

$$\alpha\beta\gamma+\alpha\beta\delta+\alpha\gamma\delta+\beta\gamma\delta=-r,$$

we obtain $\quad \Sigma\alpha^2\beta\gamma+4\alpha\beta\gamma\delta = pr;$

hence $\quad \Sigma\alpha^2\beta\gamma = pr-4s.$

6. Find for the same biquadratic the value of the symmetric function

$$\alpha^2+\beta^2+\gamma^2+\delta^2.$$

7. Find the value, in terms of the coefficients, of the sum of the squares of the roots of the equation

$$x^n+p_1x^{n-1}+p_2x^{n-2}+\cdots+p_n=0.$$

Ans. $\Sigma\alpha_1^2 = p_1^2-2p_2$.

CHAPTER VIII.

TRANSFORMATION OF EQUATIONS.

In many cases the discussion and solution of an equation is facilitated by some algebraic transformation that will change it into a form more convenient for investigation. We shall now consider some simple and useful cases of transformation.

102. To transform an Equation into Another, the Roots of which are those of the Proposed Equation with Contrary Sign.

Let $a_1, a_2, a_3 \cdots a_n$ be the roots of the equation

$$x^n + p_1x^{n-1} + p_2x^{n-2} + \cdots + p_n = 0.$$

We have then the identity

$$\begin{aligned} &x^n + p_1x^{n-1} + p_2x^{n-2} + \cdots + p_n \\ &\equiv (x - a_1)(x - a_2)(x - a_3) \cdots (x - a_n). \end{aligned}$$

Changing x into $-y$, we have, whether n be even or odd,

$$\begin{aligned} &y^n - p_1y^{n-1} + p_2y^{n-2} - \cdots \mp p_{n-1}y \pm p_n \\ &\equiv (y + a_1)(y + a_2)(y + a_3) \cdots (y + a_n) = 0. \end{aligned}$$

The roots of the last equation are $-a_1, -a_2, -a_3 \cdots -a_n$, and thus the transformed equation may be obtained from the given equation by *changing the sign of the coefficient of every other term beginning with the second.*

In applying this rule to an equation that is not complete, we must first supply the missing terms by writing them down, each in its proper place with zero for a coefficient.

EXAMPLES.

1. Find the equation whose roots are the roots of

$$x^6 - 4x^5 + 3x^4 + x^3 + 7x^2 + 2x + 5 = 0$$

with their signs changed.

2. Change the signs of the roots of the equation

$$x^6 + 2x^5 + 4x^4 + x^3 + 5x^2 + 6 = 0.$$

Ans. $x^6 + 2x^5 + 4x^4 - x^3 + 5x^2 + 6 = 0.$

103. *To transform an equation into another, the roots of which are equal to those of the proposed equation multiplied by a given quantity.*

Let $a_1, a_2, a_3, \cdots a_n$ be the roots of an equation $f(x) = 0$, and let it be required to transform the proposed equation into another, the roots of which shall be $ka_1, ka_2, ka_3, \cdots ka_n$.

Assume $x = \frac{y}{k}$, and substitute in the identity of the preceding article. After multiplying by k^n, we have

$$y^n + kp_1y^{n-1} + k^2p_2y^{n-2} + \cdots + k^{n-1}p_{n-1}y + k^np_n$$
$$\equiv (y - ka_1)(y - ka_2)\cdots(y - ka_n).$$

Hence, to multiply the roots of an equation by a given quantity k, we *have only to multiply the successive coefficients, beginning with the second, by* $k, k^2, k^3, \cdots k^n$.

Any missing power of x must be written with zero as its coefficient before the rule is applied.

This transformation is very useful for removing the coefficient of the first term when it is not unity, and, in general, for removing any fractional coefficients. When there are fractional coefficients, we get rid of them by using a multiplier k which may be determined by inspection.

EXAMPLES.

1. Change the equation

$$2x^4 - 3x^3 + 5x^2 - 4x + 6 = 0$$

into another the coefficient of whose highest term will be unity. We multiply the roots by 2.

Ans. $x^4 - 3x^3 + 10x^2 - 16x + 48 = 0.$

2. Make a similar transformation for the equation

$$3x^5 + x^4 - 5x^3 + 2x^2 - 7x + 5 = 0.$$

3. Remove the fractional coefficients from the equation

$$x^3 - \tfrac{1}{2}x^2 + \tfrac{2}{3}x - 1 = 0.$$

Here we multiply the roots by 6, thus

$$x^3 - \tfrac{1}{2}(6)x^2 + \tfrac{2}{3}(6)^2x - (6)^3 = 0.$$

Ans. $x^3 - 3x^2 + 24x - 216 = 0.$

4. Remove the fractional coefficients from the equation

$$x^4 + \tfrac{3}{10}x^2 + \tfrac{13}{25}x + \tfrac{77}{1000} = 0,$$

supply missing term, and use 10 as a multiplier.

Ans. $x^4 + 30x^2 + 520x + 770 = 0.$

Remove the fractional coefficients from the following equations:

5. $x^4 - \tfrac{2}{15}x^2 + \tfrac{3}{7}x + 1 = 0.$

6. $x^3 - \tfrac{3}{2}x^2 + \tfrac{5}{4}x - \tfrac{2}{3} = 0.$

7. $x^3 - \tfrac{1}{3}x^2 - \tfrac{1}{18}x + \tfrac{1}{108} = 0.$

8. $x^4 - \tfrac{1}{5}x^3 + \tfrac{3}{5}x^2 + \tfrac{1}{5}x - \tfrac{1}{125} = 0.$

104. To Transform an Equation into Another the Roots of which are the Reciprocals of the Roots of the Proposed Equation. Here we substitute $\frac{1}{y}$ for x in the identity of Article 102. Making this substitution and reducing, we have

$$\frac{1}{y^n}+\frac{p_1}{y^{n-1}}+\frac{p_2}{y^{n-2}}+\cdots+\frac{p_{n-1}}{y}+p_n$$

$$\equiv\frac{p_n}{y^n}\left(y-\frac{1}{a_1}\right)\left(y-\frac{1}{a_2}\right)\cdots\left(y-\frac{1}{a_n}\right),$$

or

$$y^n+\frac{p_{n-1}}{p_n}y^{n-1}+\frac{p_{n-2}}{p_n}y^{n-2}+\cdots+\frac{p_1}{p_n}y+\frac{1}{p_n}$$

$$\equiv\left(y-\frac{1}{a_1}\right)\left(y-\frac{1}{a_2}\right)\cdots\left(y-\frac{1}{a_n}\right).$$

Hence, if, in the given equation, we replace x by $\frac{1}{y}$ and multiply by y^n, the resulting equation will have for roots the reciprocals of $a_1, a_2, \cdots a_n$.

EXAMPLES.

Find the equations whose roots are the reciprocals of the roots of

1. $x^4-3x^3+7x^2+5x-2=0.$

Ans. $2y^4-5y^3-7y^2+3y-1=0.$

2. $x^5-7x^4+4x^3-7x+2=0.$
3. $x^6-5x^4-x^3+5x^2+7x+10=0.$
4. $x^3-3x^2-6=0.$

105. Infinite Roots. If $p_n=0$, one root of $f(x)=0$ is zero, and, therefore, by Art. 101, the corresponding root of

$$f\left(\frac{1}{x}\right)=0 \text{ is } \frac{1}{0}=\infty.$$

That is, *if in an equation the coefficient of* x^n (*the highest power of* x) *is 0, one root is infinity.*

Thus, one root of the equation

$$(m-n)x^3 - 3mx^2 + 2x - 10 = 0$$

is infinite, if $m = n$.

In like manner, *if the coefficients of* x^n *and* x^{n-1} *are both 0, two roots are infinity, and so on.*

106. Reciprocal Equations. *Reciprocal* or *recurring* equations are those which remain unaltered when x is changed into its reciprocal.

The conditions that must hold among the coefficients of an equation in order that it should belong to this class are, by Art. 104, as follows:

$$\frac{p_{n-1}}{p_n} = p_1, \ \frac{p_{n-2}}{p_n} = p_2, \cdots \frac{p_1}{p_n} = p_{n-1}, \ \frac{1}{p_n} = p_n.$$

The last of these conditions gives $p_n^2 = 1$, or $p_n = \pm 1$. Reciprocal equations are divided into two classes, according as p_n is equal to $+1$, or to -1.

(*a*) In the first case, we have

$$p_{n-1} = p_1, \ p_{n-2} = p_2, \ \cdots \ p_1 = p_{n-1};$$

and these relations determine the *first class of reciprocal equations*, in which the coefficients of the corresponding terms taken from the beginning and end are equal in magnitude, and have the same sign.

(*b*) In the second case, when $p_n = -1$, we have

$$p_{n-1} = -p_1, \ p_{n-2} = -p_2, \ \cdots \ p_1 = -p_{n-1};$$

which relations give the *second class of reciprocal equations*, in which corresponding terms taken from the beginning and end

are equal in magnitude, but different in sign. In this case, when the degree of the equation is even, say $n = 2m$, one of the conditions becomes $p_m = -p_m$, or $p_m = 0$, so that in reciprocal equations of the second class, whose degree is even, the middle term is absent.

It is evident that the roots of reciprocal equations occur in pairs, $\alpha, \frac{1}{\alpha}$; $\beta, \frac{1}{\beta}$; etc. When the degree is odd, there must be a root which is its own reciprocal, and it is obvious that in this case -1 or $+1$ is a root according as the equation is of the first or second class. In either case we can divide by the known factor ($x + 1$ or $x - 1$), and what is left is a reciprocal equation of even degree and of the first class.

In equations of the second class of even degree $x^2 - 1$ is a factor, and, by dividing by $x^2 - 1$, this is reducible to a reciprocal equation of the first class of even degree. Hence all reciprocal equations may be reduced to *those of the first class of even degree*, which, therefore, may be regarded as *the standard form of reciprocal equations.*

We append a few examples, with some hints as to the method of solving such equations.

EXAMPLES.

1. Solve the reciprocal equation

$$x^4 + x^3 - 4x^2 + x + 1 = 0.$$

Dividing through by x^2, this becomes

$$x^2 + x - 4 + \frac{1}{x} + \frac{1}{x^2} = 0.$$

Adding and subtracting 2, this may be put in the form

$$\left(x + \frac{1}{x}\right)^2 + \left(x + \frac{1}{x}\right) = 6,$$

and

$$\left(x + \frac{1}{x}\right)^2 + \left(x + \frac{1}{x}\right) + \frac{1}{4} = \frac{25}{4};$$

therefore $$x+\frac{1}{x}+\frac{1}{2}=\pm\frac{5}{2}.$$

$$\therefore\ x+\frac{1}{x}=2 \text{ or } -3.$$

Solving this quadratic, the first value gives $x=1$, and the second gives $x=\dfrac{-3\pm\sqrt{5}}{2}$.

2. Solve the equation
$$x^5-1=0.$$

This is a reciprocal equation of the second class. Dividing by $x-1$ (since $x=1$ is evidently a root), we reduce it to the reciprocal equation of the first class of the fourth degree.

$$x^4+x^3+x^2+x+1=0,$$

or, dividing by x^2 and arranging terms,

$$\left(x^2+\frac{1}{x^2}\right)+\left(x+\frac{1}{x}\right)=-1.$$

Therefore $$\left(x+\frac{1}{x}\right)^2+\left(x+\frac{1}{x}\right)=1.$$

Solving this as in the preceding example, we get finally

$$x=\tfrac{1}{4}\{1\mp\sqrt{5}\pm\sqrt{-1}\,(10\pm 2\sqrt{5})^{\frac{1}{2}}\},$$

which expression gives the four values of x.

3. Reduce to a reciprocal equation of even degree and of first class
$$x^6+\tfrac{5}{6}x^5-\tfrac{37}{6}x^4+\tfrac{37}{6}x^2-\tfrac{5}{6}x-1=0.$$

4. Solve the reciprocal equation
$$2x^6+x^5-13x^4+13x^2-x-2=0.$$

Divide the left-hand member by x^2-1.

107. To transform an Equation into Another, the Roots of which shall be Less (or Greater) than those of the Proposed Equation by a Constant Difference.

Let $f(x)=0$ be the proposed equation. In this equation we change x into $y+k$. The resulting equation in y will have roots each less or greater by k than the given equation in x, according as k is positive or negative. The resulting equation is (Art. 80)

$$f(k)+f'(k)y+\frac{f''(k)}{1\cdot 2}y^2+\cdots+y^n=0.$$

The following mode of formation of this equation is, for practical purposes, much more convenient than the direct calculation of the derived functions and the substitution in them of k.

Let the proposed equation be

$$f(x)=x^n+p_1x^{n-1}+p_2x^{n-2}+\cdots+p_{n-1}x+p_n=0,$$

and suppose the transformed polynomial in y to be

$$P_0y^n+P_1y^{n-1}+P_2y^{n-2}+\cdots+P_{n-1}y+P_n;$$

since $y=x-k$, this is equivalent to

$$P_0(x-k)^n+P_1(x-k)^{n-1}+\cdots+P_{n-1}(x-k)+P_n,$$

which must be identical with the given polynomial. We conclude that if the given polynomial be divided by $x-k$, the remainder is P_n, and the quotient

$$P_0(x-k)^{n-1}+P_1(x-k)^{n-2}+\cdots+P_{n-2}(x-k)+P_{n-1};$$

if this again be divided by $x-k$, the remainder is P_{n-1}, and the quotient

$$P_0(x-k)^{n-2}+P_1(x-k)^{n-3}+\cdots+P_{n-2}.$$

Proceeding in this way we can, by a repetition of the operations explained in Art. 82, calculate in succession the several

coefficients P_n, P_{n-1}, etc., of the transformed equation; the last, P_0, being equal to unity, as we know from other considerations.

We shall find, when we give in Chapter IX. an explanation of Horner's Method, that the best practical method of solving numerical equations is only an extension of the process here indicated. A few examples will make the process plain.

EXAMPLES.

1. Find the equation whose roots are the roots of

$$x^4 + x^3 - 29x^2 - 9x + 180,$$

each diminished by 6.

The calculation is best exhibited as follows:

1	1	−29	−9	180
	6	42	78	414
	7	13	69	594
	6	78	546	
	13	91	615	
	6	114		
	19	205		
	6			
	25			

Here the first division of the given polynomial by $x - 6$ gives the remainder $594\,(P_4)$, and the quotient

$$x^3 + 7x^2 + 13x + 69 \text{ (compare Art. 80).}$$

Dividing this again by $x - 6$, we get the remainder $615\,(P_3)$ and the quotient $x^2 + 13x + 91$. Dividing again, we get the remainder $205\,(P_2)$ and the quotient $x + 19$, and dividing this

we get $P_1 = 25$, and $P_0 = 1$; hence the required transformed equation is

$$y^4 + 25y^3 + 205y^2 + 615y + 594 = 0.$$

2. Find the equation whose roots are the roots of

$$x^5 + 4x^3 - x^2 + 11 = 0,$$

each diminished by 3.

Ans. $y^5 + 15y^4 + 94y^3 + 305y^2 + 507y + 353 = 0.$

3. Find the equation whose roots are the roots of

$$4x^5 - 2x^3 + 7x - 3 = 0,$$

each increased by 2.

Here we divide by $x + 2$, as follows:

$$\begin{array}{rrrrrr}
4 & 0 & -\ 2 & 0 & 7 & -\ 3 \\
 & -\ 8 & 16 & -\ 28 & 56 & -126 \\
\hline
 & -\ 8 & 14 & -\ 28 & 63 & -129 \\
 & -\ 8 & 32 & -\ 92 & 240 & \\
\hline
 & -16 & 46 & -120 & 303 & \\
 & -\ 8 & 48 & -188 & & \\
\hline
 & -24 & 94 & -308 & & \\
 & -\ 8 & 64 & & & \\
\hline
 & -32 & 158 & & & \\
 & -\ 8 & & & & \\
\hline
 & -40 & & & &
\end{array}$$

The transformed equation is therefore

$$4y^5 - 40y^4 + 158y^3 - 308y^2 + 303y - 129 = 0.$$

4. Increase by $\cdot 5$ the roots of the equation

$$3x^4 + 7x^3 - 15x^2 + x - 2 = 0.$$

5. Diminish by 20 the roots of the equation

$$5x^3 - 13x^2 - 12x + 7 = 0.$$

108. Removal of Terms. The solution of an equation is often facilitated by the removal of a certain specified term, which can be done by the transformation of Art. 107, as we shall now show.

If $f(x) = 0$ be expressed in the form

$$a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \cdots + a_n = 0,$$

and the transformed equation be written in descending powers of y, we have

$$a_0y_n + (na_0k + a_1)y^{n-1} + \left\{\frac{n(n-1)}{1 \cdot 2}a_0k^2 + (n-1)a_1k + a_2\right\}y^{n-2}$$
$$+ \cdots = 0.$$

If we give k such a value that $na_0k + a_1 = 0$, the transformed equation will be wanting in the second term.

If k be either of the values which satisfy the equation

$$\frac{n(n-1)}{1 \cdot 2}a_0k^2 + (n-1)a_1k + a_2 = 0,$$

the transformed equation will want the third term.

To remove the fourth term, a cubic equation will have to be solved; and so on. The following examples will illustrate the method:

EXAMPLES.

1. Transform the equation

$$x^3 - 6x^2 + 12x + 19 = 0$$

into one wanting the second term.

$na_0k + a_1 = 0$ gives $k = 2$; therefore we must diminish the roots by 2. *Ans.* $y^3 + 27 = 0$.

2. Transform the equation

$$x^4 - 4x^3 - 18x^2 - 3x + 2 = 0$$

into one wanting the third term.

The quadratic for k is

$$6k^2 - 12k - 18 = 0, \text{ giving } k = 3,\ k = -1.$$

Thus there are two ways of effecting the transformation. Diminishing the roots by 3, we get

$$y^4 + 8y^3 - 111y - 196 = 0.$$

Increasing the roots by 1, we get

$$y^4 - 8y^3 + 17y - 8 = 0.$$

3. Transform the equation

$$x^4 + 8x^3 + x - 5 = 0$$

into one wanting the second term.

4. Transform the equation

$$x^3 - 6x^2 + 9x - 10 = 0$$

into one wanting the third term.

109. The Algebraic Solution of the Cubic Equation. Let the general cubic equation be written in the form

$$x^3 + 3p_1x^2 + 3p_2x + p_3 = 0 \quad . \quad . \quad . \quad . \quad . \quad (1)$$

We first simplify this by transforming it into an equation lacking the second term. To do this, we replace x by $y + k$ (Art. 107), where k is determined by the equation (Art. 108)

$$3k + 3p_1 = 0,$$

which gives $k = -p$.

Then (1) becomes

$$(y - p_1)^3 + 3p_1(y - p_1)^2 + 3p_2(y - p_1) + p_3 = 0 \quad . \quad (2)$$

which reduces to the form

$$y^3 + 3Hy + G = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

where $\qquad H \equiv p_2 - p_1^2$ and $G \equiv 2p_1^3 - 3p_1p_2 + p_3$.

To solve (3), assume

$$y = r^{\frac{1}{3}} + s^{\frac{1}{3}};$$

$$\therefore\ y^3 = r + s + 3 r^{\frac{1}{3}} s^{\frac{1}{3}} (r^{\frac{1}{3}} + s^{\frac{1}{3}}).$$

$$\therefore\ y^3 - 3 r^{\frac{1}{3}} s^{\frac{1}{3}} y - (r + s) = 0 \quad . \quad . \quad . \quad . \quad . \quad (4)$$

Comparing coefficients in (3) and (4), we have

$$r^{\frac{1}{3}} s^{\frac{1}{3}} = -H,\ r + s = -G;$$

from which equations we obtain

$$r = \tfrac{1}{2}(-G + \sqrt{G^2 + 4H^3}) \quad . \quad . \quad . \quad . \quad . \quad (5)$$

$$s = \tfrac{1}{2}(-G - \sqrt{G^2 + 4H^3}) \quad . \quad . \quad . \quad . \quad . \quad (6)$$

and, substituting for $s^{\frac{1}{3}}$ its value $\dfrac{-H}{r^{\frac{1}{3}}}$, we have

$$y = r^{\frac{1}{3}} + \frac{-H}{r^{\frac{1}{3}}}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

the value of r being given in (5).

We observe that if r be replaced by s, this value of y is unchanged, as the terms are then simply interchanged; also, since $r^{\frac{1}{3}}$ has the three values $\sqrt[3]{r}$, $\omega\sqrt[3]{r}$, $\omega^2\sqrt[3]{r}$, obtained by multiplying any one of its values by the three cube roots of unity (Art. 100), we obtain three, and only three, values for y; namely,

$$\sqrt[3]{r} + \frac{-H}{\sqrt[3]{r}},\ \omega\sqrt[3]{r} + \frac{-H}{\omega\sqrt[3]{r}},\ \omega^2\sqrt[3]{r} + \frac{-H}{\omega^2\sqrt[3]{r}}.$$

We have then $\quad x + p_1 = r^{\frac{1}{3}} + \dfrac{-H}{r^{\frac{1}{3}}} \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$

as the *complete algebraic solution* of the cubic equation

$$x^3 + 3p_1x^2 + 3p_2x + p_3 = 0,$$

the square root and cube root involved being taken in their entire generality.*

* This solution is known as *Cardan's Solution*, because it was first published by him in 1545. See Historical Note, page 76.

110. Application to Numerical Equations. The solution of the cubic obtained in the last article is of little practical value, when the equation has numerical coefficients. For, when the roots are all *real* and *unequal*, $G^2 + 4H^3 < 0$ (this may be shown by Sturm's Theorem, see Chapter IX), whence r is imaginary, and the roots involve the square root of an imaginary number, which in general we cannot solve. If the equation has equal roots, it can be solved; and if it has a pair of imaginary roots, it likewise can be solved, for in this case $G^2 + 4H^3$ is positive. In the first case, namely, when the roots are all real, the roots may be computed by the use of Trigonometry.*

To illustrate this method by an example, let us solve the equation

$$x^3 - 18x - 35 = 0. \quad \ldots \ldots \quad (1)$$

Put
$$x = r^{\frac{1}{3}} + s^{\frac{1}{3}};$$

$$\therefore\ x^3 - 3r^{\frac{1}{3}}s^{\frac{1}{3}}x - (r+s) = 0 \quad \ldots \ldots \quad (2)$$

$$\therefore\ r^{\frac{1}{3}}s^{\frac{1}{3}} = 6,\ r + s = 35,$$

$$r^{\frac{1}{3}} = 3,\ s^{\frac{1}{3}} = 2;$$

$$\therefore\ x = r^{\frac{1}{3}} + s^{\frac{1}{3}} = 3 + 2 = 5.$$

The other two roots are

$$\omega\sqrt[3]{r} + \frac{-H}{\omega\sqrt[3]{r}} = -\tfrac{5}{2} + \tfrac{1}{2}\sqrt{-3},$$

$$\omega^2\sqrt[3]{r} + \frac{-H}{\omega^2\sqrt[3]{r}} = -\tfrac{5}{2} - \tfrac{1}{2}\sqrt{-3}.$$

* Throughout a treatise of the grade and scope of this work, there is obviously much matter that must be left unnoticed. It would be interesting to give some trigonometrical solutions of the cubic and bi-quadratic, and the author reluctantly dismisses the subject by referring the student to more extended treatises on the Theory of Equations.

After getting the real root, it is often simpler to depress the equation and then get the two imaginary roots by solving the resulting quadratic. Here the depressed equation is

$$x^2 + 5x + 7 = 0 \; . \; . \; . \; . \; . \; . \; . \quad (3)$$

and the roots of this quadratic are

$$-\tfrac{5}{2} + \tfrac{1}{2}\sqrt{-3} \text{ and } -\tfrac{5}{2} - \tfrac{1}{2}\sqrt{-3},$$

which agrees with what we have just obtained.

EXAMPLES.

Solve the following equations:

1. $x^3 - 6x^2 + 10x = 8.$ *Ans.* $4, 1 + \sqrt{-1}, 1 - \sqrt{-1}.$
2. $x^3 - 9x^2 + 28x = 30.$ *Ans.* $3, 3 + \sqrt{-1}, 3 - \sqrt{-1}.$
3. $x^3 + 72x = 1720.$
4. $x^3 + 63x = 316.$
5. $x^3 - 6x^2 + 13x = 10.$
6. $x^3 - 6x^2 + 3x - 18.$

111. Solution of the Biquadratic Equation.*

Here we find it convenient to put the biquadratic in the form

$$x^4 + 2px^3 + qx^2 + 2rx + s = 0 \; . \; . \; . \; . \; . \quad (1)$$

Adding $(ax + b)^2$ to both members, we obtain

$$x^4 + 2px^3 + (q + a^2)x^2 + 2(r + ab)x + s + b^2 = (ax + b)^2. \quad . \quad (2)$$

Assume

$$x^4 + 2px^3 + (q + a^2)x^2 + 2(r + ab)x + s + b^2 \equiv (x^2 + px + k)^2. \quad . \quad (3)$$

Equating coefficients, we have

$$p^2 + 2k = q + a^2, \; pk = r + ab, \; k^2 = s + b^2. \quad . \quad . \quad (4)$$

* The solution here given is due to Ferrari. (See Historical Note, page 76.) This and the solutions of Descartes, Euler, Laplace, Lagrange, and others, all involve the solution of a cubic by Cardan's method, and will of course fail when that fails. We would then employ a trigonometrical solution.

Eliminating a and b from (4), we have

$$(pk - r)^2 = (2k + p^2 - q)(k^2 - s),$$

or $$2k^3 - qk^2 + 2(pr - s)k - p^2s + qs - r^2 = 0\,. \quad . \quad . \quad (5)$$

From this cubic we find, if possible, a real value of k by the method of Art. 109. The values of a and b are then known from (4).

Subtracting (2) from (3), we have

$$(x^2 + px + k)^2 - (ax + b)^2 = 0,$$

which is equivalent to the two quadratic equations

$$x^2 + (p - a)x + (k - b) = 0,$$

$$x^2 + (p + a)x + (k + b) = 0,$$

the roots of which are readily obtained.

As an example of this method, let us solve the equation

$$x^4 + 2x^3 - 7x^2 - 8x + 12 = 0\,. \quad . \quad . \quad . \quad . \quad (1)$$

Adding $(ax + b)^2$ to both members, we obtain

$$x^4 + 2x^3 + (a^2 - 7)x^2 + 2(ab - 4)x + b^2 + 12 = (ax + b)^2\,. \quad . \quad (2)$$

Since $p = +1$, assume

$$x^4 + 2x^3 + (a^2 - 7)x^2 + 2(ab - 4)x + b^2 + 12 \equiv (x^2 + x + k)^2 \quad . \quad (3)$$

Equating coefficients, we have

$$a^2 - 7 = 2k + 1,\; ab - 4 = k,\; b^2 + 12 = k^2 \quad . \quad . \quad (4)$$

$$\therefore\ (2k + 8)(k^2 - 12) = (k + 4)^2,$$

$$\therefore\ 2k^3 + 7k^2 - 32k - 112 = 0.$$

Whence $k = 4$; hence $a^2 = 16$, $ab = 8$, $b^2 = 4$, and $\therefore\ a = 4$, $b = 2$.

Therefore, from (2), (3), and (4), we obtain

$$(x^2 + x + 4)^2 - (4x + 2)^2 = 0,$$

which is equivalent to the two equations

$$x^2 - 3x + 2 = 0, \quad x^2 + 5x + 6 = 0;$$

and, therefore, the four roots are 1, 2, -2, -3.

EXAMPLES.

1. Solve $x^4 - 6x^3 + 12x^2 - 14x + 3 = 0$.

2. Solve $x^4 + 4x^3 + 3x^2 - 44x - 84 = 0$.

3. Solve $x^4 - 6x^2 - 8x - 3 = 0$. *Ans.* $-1, -1, -1, 3$.

4. Solve $x^4 - 3x^2 - 42x - 40 = 0$.

CHAPTER IX.

LIMITS OF THE ROOTS OF AN EQUATION.

112. Definition of Limits. In attempting to find the real roots of numerical equations, it is very advantageous to narrow the limits within which such roots must be sought. Descartes' Rule of Signs gives us the limit of the number of real roots, but tells us nothing as to the limit of the value of such roots. The closing remarks of Art. 78 suggest that there are means of getting the limits between which the roots of a given equation must lie, and we shall now proceed to give some of the methods for doing this.

A *superior limit* of the *positive* roots is any positive number greater than the greatest of the roots, that is, nearer $+\infty$; an *inferior limit* of the positive roots is any positive number smaller than the smallest of them.

A *superior limit* of the *negative* roots is any negative number greater in absolute value than the greatest of them, that is, nearer to $-\infty$ than the greatest; an *inferior limit* of the negative roots is any negative number smaller in absolute value than the smallest of them. In the next three articles we have three rules for the determination of the superior limits of the positive roots.

113. PROPOSITION I. *In an equation*

$$f(x) \equiv x^n + p_1x^{n-1} + p_2x^{n-2} + \cdots + p_{n-1}x + p_n = 0,$$

if the first negative term be $-p_rx^{n-r}$, *and if the greatest negative coefficient be* $-p_k$, then $\sqrt[r]{p_k}+1$ *is a superior limit of the positive roots.*

Now $f(x)$ is certainly positive for any value of x, which makes

$$x^n > p_k(x^{n-r} + x^{n-r+1} + \cdots + x + 1) > p_k \frac{x^{n-r+1} - 1}{x - 1}.$$

But this inequality is true, taking $x > 1$, if

$$x^n > p_k \frac{x^{n-r+1}}{x-1},$$

or $$x^{n+1} - x^n > p_k x^{n-r+1},$$

or $$x - 1 > p_k x^{-r+1},$$

or $$x^r - x^{r-1} > p_k,$$

that is $$x^{r-1}(x-1) > p_k.$$

But, since $$x^{r-1} > (x-1)^{r-1},$$

$$x^{r-1}(x-1) \text{ is } > p_k, \text{ if } (x-1)^{r-1}(x-1) > p_k,$$

or $$(x-1)^r > p_k.$$

Hence $f(x)$ will always be positive, if $x =$ or $> 1 + \sqrt[r]{p_k}$.

Hence $\sqrt[r]{p_k} + 1$ is a superior limit of the positive roots.

114. Proposition II. *If in any equation each negative coefficient be taken positively, and divided by the sum of all the positive coefficients which precede it, the greatest quotient thus formed increased by unity is a superior limit of the positive roots.*

Let the equation be

$$a_0x^n + a_1x^{n-1} + a_2x^{n-2} - a_3x^{n-3} + \cdots - a_rx^{n-r} + \cdots + a_n = 0 \quad (1)$$

in which we regard the fourth coefficient as negative, and we consider also a general negative coefficient; namely, $-a_r$.

Now, since

$$\frac{x^n - 1}{x - 1} = x^{n-1} + x^{n-2} + \cdots + x + 1,$$

we have $x^n = (x-1)(x^{n-1} + x^{n-2} + \cdots + x + 1) + 1.$

Let us now develop each positive term of equation (1) by the formula

$$a_m x^m = a_m(x-1)(x^{m-1} + x^{m-2} + \cdots + x + 1) + a_m,$$

the negative term remaining unchanged.

The polynomial $f(x)$ becomes then: $a_0(x-1)x^{n-1}$

$$\begin{aligned}
&+ a_0(x-1)x^{n-2} + a_0(x-1)x^{n-3} + \cdots + a_0(x-1)x^{n-r} + \cdots + a_0, \\
&+ a_1(x-1)x^{n-2} + a_1(x-1)x^{n-3} + \cdots + a_1(x-1)x^{n-r} + \cdots + a_1, \\
&\qquad + a_2(x-1)x^{n-3} + \cdots + a_2(x-1)x^{n-r} + \cdots + a_2, \\
&\qquad - a_3 x^{n-3}, \\
&\qquad\qquad + \cdots\cdots\cdots \\
&\qquad\qquad\qquad - a_r x^{n-r}, \\
&\qquad\qquad\qquad\quad + \cdots\cdots \\
&\qquad\qquad\qquad\qquad\qquad + a_n.
\end{aligned}$$

In the new polynomial thus formed, representing the left-hand member of the transformed equation, the successive coefficients of x^{n-1}, x^{n-2}, etc., are

$$a_0(x-1),\ (a_0 + a_1)(x-1),\ (a_0 + a_1 + a_2)(x-1) - a_3, \text{ etc.}$$

Any value of x greater than unity is sufficient to make positive every term in which no negative coefficient a_3, a_r, etc., occurs. To make the latter terms positive, we must have

$$(a_0 + a_1 + a_2)(x-1) > a_3,$$

$$\cdots\cdots\cdots\cdots\cdots$$

$$(a_0 + a_1 + a_2 + \cdots a_{r-1})(x-1) > a_r, \text{ etc.}$$

Hence $$x > \frac{a_3}{a_0 + a_1 + a_2} + 1 \cdots$$

$$x > \frac{a_r}{a_0 + a_1 + a_2 + \cdots + a_{r-1}} + 1, \text{ etc.}$$

If now we take for x the greatest of all these quantities, the first member will be positive (for this value and for all greater values of x); and this will be a superior limit of the roots.

115. Limit obtained by grouping Certain Terms. It is usually possible to determine, by inspection, a limit closer than that given by either of the preceding propositions. In this method we arrange the terms of an equation in groups having a positive term first, and then observe what is the lowest integral value of x, which will have the effect of rendering each group positive. Such a value of x will be a superior limit of the roots.

The form of the equation will suggest the arrangement into groups in each case.

Of the propositions in the two preceding articles, sometimes one will give the closer limit, sometimes the other. In most cases Prop. II will give the closer limit. Of course the smaller the number found, the better. We consider the integer next above the numerical value found by either rule as the limit.

EXAMPLES.

1. Find a superior limit of the positive roots of the equation

$$x^4 - 5x^3 + 40x^2 - 8x + 23 = 0.$$

Art. 113 gives $8 + 1$, or 9, as a limit, Art. 114 gives $\frac{5}{1} + 1$, or 6, as a limit.

Hence 6 is a superior limit.

2. Find a superior limit of the positive roots of

$$x^7 + 4x^6 - 3x^5 + 5x^4 - 9x^3 - 11x^2 + 6x - 8 = 0.$$

Art. 113 gives 5 as limit.

Of the fractions

$$\frac{3}{1+4}, \ \frac{9}{1+4+5}, \ \frac{11}{1+4+5}, \ \frac{8}{1+4+5+6},$$

the third is the greatest, and Art. 114 gives the limit 3. In this case Art. 114 gives the closer limit.

3. Find the superior limit of the positive roots of

$$x^5 + 8x^4 - 14x^3 - 53x^2 + 56x - 18 = 0.$$

Here, Art. 113 gives 9 as a limit, and Art. 114 gives 7 as a limit.

4. Find the superior limit of the positive roots of

$$x^8 + 20x^7 + 4x^6 - 11x^5 - 120x^4 + 13x - 25 = 0.$$

The methods of Arts. 113, 114 both give the limit 6.

In this case we can find a much closer limit by applying the method of Art. 115.

The equation may be arranged as follows:

$$x^5(x^3 - 11) + 20x^4(x^3 - 6) + 4x^6 + 13x - 25 = 0.$$

Here $x = 3$, or any greater number, renders each group positive; hence 3 is a limit.

5. Find a superior limit of the roots of the equation

$$x^4 - x^3 - 2x^2 - 4x - 24 = 0.$$

When there are several negative terms, and the coefficient of the highest term is unity, it is convenient to multiply the whole equation by such a number as will enable us to distribute the highest term among the negative terms. Here, multiplying by 4, we can write the equation as follows:

$$x^3(x - 4) + x^2(x^2 - 8) + x(x^3 - 16) + x^4 - 96 = 0,$$

and 4 is a superior limit.

Find a superior limit of the positive roots of the following equations:

6. $4x^5 - 8x^4 + 22x^3 + 98x^2 - 73x + 5 = 0.$

7. $5x^5 - 7x^4 - 10x^3 - 23x^2 - 90x - 317 = 0.$

8. $x^5 - x^4 - 2x^3 + 2x^2 + x - 1 = 0.$

9. $x^4 - 8x^3 + 12x^2 + 16x - 39 = 0.$

116. Inferior Limits, and Limits of the Negative Roots.

To find an inferior limit of the positive roots, we must transform the equation into another whose roots are the reciprocals of those of the first by the substitution $x=\frac{1}{y}$ (Art. 104).

Find then the superior limit l of the positive roots of the equation in y. The reciprocal of this, $\frac{1}{l}$, will be the required inferior limit; for since $y<l$, $\frac{1}{y}>\frac{1}{l}$, *i.e.*, $x>\frac{1}{l}$.

For example, take the equation of example (3) under the last article

$$x^5+8x^4-14x^3-53x^2+56x-18=0. \quad . \quad . \quad (1)$$

Putting $x=\frac{1}{y}$, (1) becomes

$$y^5-\tfrac{56}{18}y^4+\tfrac{53}{18}y^3+\tfrac{14}{18}y^2-\tfrac{8}{18}y-\tfrac{1}{18}=0, \quad . \quad . \quad (2)$$

and a superior limit of (2), by Art. 114, is $\frac{56}{18}+1=\frac{74}{18}$, and, therefore, $\frac{18}{74}$ is an inferior limit of the positive roots.

To find limits of the negative roots, we have only to transform the equation by the substitution $x=-y$.

This transformation (Art. 102) changes the negative into positive roots. If l and l' be the superior and inferior limits of the positive roots of the equation in y, then $-l$ and $-l'$ are the limits of the negative roots of the proposed equation.

For example, take the equation

$$x^4-2x^3-13x^2-14x+24=0. \quad . \quad . \quad . \quad (1)$$

Putting $x=-y$, this becomes

$$y^4+2y^3-13y^2+14y+24=0. \quad . \quad . \quad . \quad (2)$$

By the method of Art. 115, we readily find a superior limit of the positive roots of (2) to be 5; therefore -5 is a superior limit of the negative roots of equation (1).

EXAMPLES.

1. Find limits to the positive and negative roots of

$$x^6 - 5x^5 + x^4 + 12x^3 - 12x^2 + 1 = 0.$$

Show that the real roots of the following equations lie between the limits respectively given:

2. $x^4 - x^3 + 4x^2 - 3x + 1 = 0$; $\frac{1}{4}$ and 1.

3. $x^4 + x^3 - 10x^2 - x + 15 = 0$; -4 and 3.

4. $x^5 + 5x^4 + x^3 - 16x^2 - 20x - 16 = 0$; -5 and 3.

5. $(x^2 - 4x - 2)^2 - 43 = 0$; -2 and 6.

6. $x^5 + 2x^4 + 3x^3 + 4x^2 + 5x - 54321$; -3, 9.

Separation of the Roots of Equations.

117. Having found the limits within which the real roots of an equation lie, the next step in the solution of an equation is to discover the intervals in which the separate roots lie. The two most useful theorems for determining the number of real roots between any two arbitrarily assumed values of the variable are the *Theorem of Fourier and Budan*, and the *Theorem of Sturm*.

For a proof of the first, we refer the reader to *Burnside and Panton's Theory of Equations*. The theorem of Sturm,* which we shall consider in the next article, has the advantage of being unfailing in its application, giving always the *exact number* of real roots between any two proposed quantities;

* J. C. F. Sturm (1803–1855).

whereas the theorem of Fourier and Budan gives only a certain limit which the number of real roots in the proposed equation cannot exceed.

118. Sturm's Theorem. Let

$$f(x) \equiv x^n + p_1x^{n-1} + \cdots + p_{n-1}x + p_n = 0 \; . \quad . \quad . \quad (1)$$

be an equation from which the multiple roots have been removed (Art. 98).*

To find the equal roots we have employed the common operation of finding the H. C. F. of a polynomial $f(x)$, and its first derived function, $f'(x)$. Sturm has employed the same operation for forming the auxiliary functions which are used in this method for separating the roots of an equation.

Let the process of finding the H. C. F. of $f(x)$ and $f'(x)$ be performed.

The successive remainders will go on diminishing in degree, and, as $f(x)$ has, by hypothesis, no multiple roots, $f(x)$ and $f'(x)$ have no common divisor except unity, and we finally obtain a remainder, $f_n(x)$, independent of x; that is, which is numerical.

Dividing $f(x)$ by $f'(x)$, we shall obtain a quotient q_1, with a remainder of a degree lower than that of $f(x)$. Denote this remainder, *with its sign changed*, by $f_2(x)$, and divide $f'(x)$ by $f_2(x)$, and so on; the operation being precisely the same as that of finding the H. C. F. of $f(x)$ and $f'(x)$, except that the signs of each remainder *must* be changed, while no other changes of sign are permissible. In the process of finding $f_2(x)$, $f_3(x)$, etc., any *positive* numerical factor may be omitted or introduced, in order to avoid fractions, for the *sign* of the result is not affected thereby.

* This limitation is not necessary, but for simplicity we consider the equation cleared of equal roots, as this can always be done by the method of Art. 98.

The expressions $f(x)$, $f'(x)$, $f_2(x)$, $f_3(x) \cdots f_n(x)$ are called *Sturm's Functions.*

Keeping in mind the above explanations and definitions, we may now state Sturm's Theorem:

THEOREM. *If any two real numbers a and b be substituted for x in Sturm's Functions*

$$f(x),\ f'(x),\ f_2(x) \cdots f_{n-1}(x),\ f_n(x),$$

and the signs noted, the difference between the number of changes of sign in the series when a is substituted for x, and the number when b is substituted for x, expresses exactly the number of real roots of the equation $f(x) = 0$ between a and b.

From the way in which Sturm's Functions are formed, we derive the following series of equations, in which

$$q_1,\ q_2,\ q_3 \cdots q_{n-1}$$

represent the successive quotients in the operation:

$$\left.\begin{array}{l} f(x) = q_1 f'(x) - f_2(x) \\ f'(x) = q_2 f_2(x) - f_3(x) \\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \\ f_{r-1}(x) = q_r f_r(x) - f_{r+1}(x) \\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \\ f_{n-2}(x) = q_{n-1} f_{n-1}(x) - f_n(x) \end{array}\right\}$$

Having regard to these relations, we observe:

(1) The last of the functions $f_n(x)$ is not zero; for by supposition it is independent of x, and if it were zero, the equation $f(x) = 0$ would have equal roots by Art. 98, which is contrary to the hypothesis.

(2) No two consecutive functions in the series can have a common factor; for, if they could, all the succeeding functions would ~~vanish~~, ~~including $f_n(x)$~~, and this is impossible by (1).

have a common factor and $f_n(x)$ would

(3) When any auxiliary function vanishes, the two adjacent functions have contrary signs. Suppose, for example, that $f_2(x) = 0$, then from the second of the above system of relations we have $f_1(x) = -f_3(x)$.

In examining, therefore, what changes of sign can take place in the series during the passage of x from a to b, we may exclude the case of two consecutive functions vanishing for the same value of the variable; therefore the different cases in which any change of sign can take place are the following:

(*a*) When x passes through a root of the equation $f(x) = 0$.

(*b*) When x passes through a value which causes one of the functions f', f_2, $f_3 \cdots f_{n-1}$ to vanish.

(*c*) When x passes through a value which causes two or more of the functions f', f_2, $f_3 \cdots f_{n-1}$ to vanish together; no two of the vanishing functions, however, being consecutive.

(*a*) When x passes through a root of $f(x) = 0$, it follows from Art. 99 that one change of sign is lost, since immediately before the passage $f(x)$ and $f'(x)$ have unlike signs, and immediately after the passage they have like signs.

(*b*) Suppose x to take a value α which is a root of the equation $f_r(x) = 0$. From the equation

$$f_{r-1}(x) = q_r f_r(x) - f_{r+1}(x),$$

we have

$$f_{r-1}(\alpha) = -f_{r+1}(\alpha),$$

which proves, as we have seen, that this value of x gives to $f_{r-1}(x)$ and $f_{r+1}(x)$ the same numerical value with different signs. In passing from a value a little less than α to one a little greater, we can suppose the interval so small that it contains no root of $f_{r-1}(x)$ or $f_{r+1}(x)$; hence, throughout the interval under consideration, these two functions retain their signs. We conclude that just before x, varying continuously, reaches the value α, the signs of $f_{r-1}(x)$, $f_r(x)$, $f_{r+1}(x)$ must be $+ \pm -$ or $- \pm +$, and just afterwards they must be $+ \mp -$ or $- \mp +$; that is, $f_r(x)$ changes sign as x passes through the

value a, and the other two do not. But, though the sign of $f_r(x)$ changes, no variation of sign is either lost or gained thereby in the group of three; because, on account of the difference of signs of the two extremes $f_{r-1}(x)$ and $f_{r+1}(x)$, there will exist both before and after the passage one variation and one permanency of sign, whatever be the sign of the middle function. For in the change from $+ \pm -$ to $+ \mp -$, or from $- \pm +$ to $- \mp +$, a permanency and a variation are changed into a variation and a permanency, or a variation and a permanency into a permanency and a variation; but no variation of sign is lost or gained on the whole.

(c) It follows at once that if two or more of the auxiliary functions vanish for the same value of x, since no two adjacent ones can vanish, the same reasoning that was employed in (b) holds good here, and, therefore, if $f(x)$ is one of the vanishing functions, one change of sign is lost, and, if not, no change is either lost or gained. We have proved, therefore, that when x passes through a root of $f(x) = 0$, one change of sign is lost, and under no other circumstances is a change either lost or gained. Hence the theorem: the number of changes of sign lost while x varies from a to b is equal to the number of real roots of the equation between a and b.

119. Separation of the Real Roots. The substitution of $+\infty$ and $-\infty$ for x in Sturm's Functions determines the number of real roots of $f(x) = 0$.

The number of imaginary roots would, of course, be the difference between the degree of the equation and the number of real roots thus determined. The substitution of $+\infty$ and 0 for x determines the number of positive real roots, and the substitution of $-\infty$ and 0 determines the number of negative real roots.

In applying Sturm's theorem, it is convenient in practice to substitute first $-\infty$, 0, $+\infty$ in Sturm's Functions, so as to obtain the whole number of negative and of positive roots.

To separate the negative roots, the integers -1, -2, -3, etc., are to be substituted in succession till we reach the same series of signs as results from the substitution of $-\infty$; and to separate the positive roots we substitute 1, 2, 3, etc., till the signs furnished by $+\infty$ are reached. A few examples will illustrate the application of the theorem.

EXAMPLES.

1. Find the number and situation of the real roots of the equation

$$f(x) \equiv x^3 - 2x - 5 = 0.$$

We find $f'(x) = 3x^2 - 2$, $f_2(x) = 4x + 15$, $f_3(x) = -643$.

Corresponding to the values $-\infty$, 0 $+\infty$ of x, we have

$$\begin{array}{rcccc} (-\infty) & - & + & - & - \\ (0) & - & - & + & - \\ (+\infty) & + & + & + & - \end{array}$$

Hence there is only one real root, and it is positive.

Again, corresponding to values, 1, 2, 3 of x, we have

$$\begin{array}{rcccc} (1) & - & + & + & - \\ (2) & - & + & + & - \\ (3) & + & + & + & - \end{array}$$

The real root, therefore, lies between (2) and (3).

2. Find the number and situation of the real roots of the equation

$$f(x) \equiv x^4 - 6x^3 + 5x^2 + 14x - 4 = 0.$$

Here $f'(x) = 2x^3 - 9x^2 + 5x + 7$, omitting a factor 2.

$$f_2(x) = 17x^2 - 57x - 5,$$

$$f_3(x) = 152x - 457,$$

$$f_4(x) = +.$$

In this example it will be found that the calculation of $f_4(x)$ is somewhat complicated; it is sufficient for our purpose,

however, to know the *sign*, and thus when we ascertain that it is *positive* we need not calculate it exactly, but merely put down $f_4(x) = +$. Here we have the following series of signs:

$$\begin{array}{rccccc} (-\infty) & + & - & + & - & + \\ (0) & - & + & - & - & + \\ (+\infty) & + & + & + & + & + \end{array}$$

Hence all the roots are real: one negative and three positive. We have further the series of signs:

$$\begin{array}{rcccccl} (-2) & + & - & + & - & +, & 4 \text{ variations.} \\ (-1) & - & - & + & - & +, & 3 \text{ variations.} \\ (0) & - & + & - & - & +, & 3 \text{ variations.} \\ (1) & + & + & - & - & +, & 2 \text{ variations.} \\ (2) & + & - & - & - & +, & 2 \text{ variations.} \\ (3) & + & - & - & - & +, & 2 \text{ variations.} \\ (4) & + & + & + & + & +, & 0 \text{ variations.} \end{array}$$

There is one change of sign lost between -2 and -1, one between 0 and 1, and two between 3 and 4.

If we put $3\frac{1}{2}$ for x, the succession of signs is $-\ 0\ +\ +\ +$, and thus there is only one change of sign, so that one root of the equation lies between 3 and $3\frac{1}{2}$; therefore another root lies between $3\frac{1}{2}$ and 4.

Find the number and situation of the real roots of the equations:

3. $x^3 - 3x^2 - 4x + 13 = 0$. **4.** $x^3 - 7x + 7 = 0$.

5. $x^4 - 4x^3 - 3x + 23 = 0$.

Ans. Two real positive roots, between 2 and 3, and 3 and 4, respectively.

6. $x^4 - 4x^3 + x^2 + 6x + 2 = 0$. **7.** $x^4 + x^3 + x - 1 = 0$.

8. $x^3 - 6x^2 + 8x + 40 = 0$.

CHAPTER X.

ELIMINATION.

120. Under the head of Applications of Determinants, in Chapter III, we have considered, as the student will recall, several cases of elimination whereby a system of equations may be solved.

In Art. 41 there was given the method of solving a system of simultaneous equations where the number of unknown quantities is the same as the number of equations.

In Arts. 42 and 43, the case where the number of equations is greater than the number of unknowns was considered, and the *condition of consistency* of such a system was obtained. In such a case the *eliminant*, or *resultant*, which is the determinant obtained by eliminating the unknowns from the given equation, is the determinant of the *coefficients* and *absolute terms*.

We next considered homogeneous linear equations (Art. 44), and found that for a system of n homogeneous linear equations involving n unknowns the *eliminant* is the *determinant of the coefficients*, and that if this determinant vanishes, the ratios of the unknowns may be determined, but not their absolute values.

There are various ways of determining the resultant, R, of a system of equations. We shall give some of the best methods of eliminating a single unknown from two consistent equations of any degree.

121. The method that naturally presents itself is as follows: The resultant of two linear equations

$$ax + b = 0,\ a'x + b' = 0$$

is evidently

$$ab' - ba' = 0.$$

If now we have two quadratic equations

$$ax^2 + bx + c = 0 \quad . \quad . \quad (1) \qquad a'x^2 + b'x + c' = 0 \quad . \quad . \quad (2)$$

multiplying the first by a', the second by a, and subtracting, we get

$$(ab')x + (ac') = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

where $(ab') \equiv \begin{vmatrix} a & b \\ a' & b' \end{vmatrix}$ and $(ac') \equiv \begin{vmatrix} a & c \\ a' & c' \end{vmatrix}$ [See Art. 17, (3)],

and, again, multiplying the first by c', the second by c, subtracting, and dividing by x, we get

$$(ac')x + (bc') = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

The problem is now reduced to elimination between two linear equations, and the result is

$$(ac')^2 + (ba')(bc') = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

This method of forming the resultant is practically very limited in application, as it becomes very tedious for equations higher than the fourth degree.

122. Euler's Method of Elimination. Having given two equations of the mth and nth degrees respectively,

$$\left.\begin{aligned} f(x) &\equiv a_0x^m + a_1x^{m-1} + \cdots + a_m = 0 \\ F(x) &\equiv b_0x^n + b_1x^{n-1} + \cdots + b_n = 0 \end{aligned}\right\} \quad . \quad . \quad . \quad (1)$$

we propose to eliminate x, or to find their resultant.

If these equations admit a common root r, we may assume

$$f(x) \equiv (x - r)f_1(x),$$

$$F(x) \equiv (x - r)F_1(x),$$

where
$$\left.\begin{aligned} f_1(x) &\equiv \alpha_1x^{m-1} + \alpha_2x^{m-2} + \cdots + \alpha_m, \\ F_1(x) &\equiv \beta_1x^{n-1} + \beta_2x^{n-2} + \cdots + \beta_n, \end{aligned}\right\} \quad . \quad . \quad . \quad (2)$$

the coefficients being undetermined quantities depending on r.

Whence we have

$$f(x)F_1(x) \equiv F(x)f_1(x),$$

an identical equation of the $(m+n-1)$th degree. Now, equating the coefficients of like powers of x on both sides of the equation, we have $m+n$ homogeneous equations of the first degree in the $m+n$ quantities $p_1, p_2, \cdots p_m, q_1, q_2, \cdots q_n$; and eliminating these quantities by the method of Art. 44, we obtain the resultant of the two given equations in the form of a determinant. The method will be made clear by a few examples.

EXAMPLES.

1. Find the resultant of the two equations

$$ax^2 + bx + c = 0, \quad a_1x^2 + b_1x + c_1 = 0,$$

supposing them to have a common root. We have identically

$$(q_1x + q_2)(ax^2 + bx + c) \equiv (p_1x + p_2)(a_1x^2 + b_1x + c_1),$$

or

$$(q_1a - p_1a_1)x^3 + (q_1b + q_2a - p_1b_1 - p_2a_1)x^2$$
$$+ (q_1c + q_2b - p_1c_1 - p_2b_1)x + q_2c - p_2c_1 \equiv 0.$$

Equating to zero all the coefficients of this equation, we have the four homogeneous equations

$$\begin{aligned} q_1a \qquad\quad - p_1a_1 \qquad &= 0, \\ q_1b + q_2a - p_1b_1 - p_2a_1 &= 0, \\ q_1c + q_2b - p_1c_1 - p_2b_1 &= 0, \\ q_2c \qquad\quad - p_2c_1 &= 0, \end{aligned}$$

and, eliminating p_1, p_2, q_1, q_2, we obtain the resultant in the form

$$\begin{vmatrix} a & 0 & a_1 & 0 \\ b & a & b_1 & a_1 \\ c & b & c_1 & b_1 \\ 0 & c & 0 & c_1 \end{vmatrix} = 0.$$

The student can easily verify that this result is the same as that of Art. 121.

2. Find the resultant of the equations

$$a_0x^3 + a_1x^2 + a_2x + a_3 = 0, \quad b_0x^2 + b_1x + b_2 = 0.$$

Enler's identity

$$(a_0x^3 + a_1x^2 + a_2x + a_3)(\beta_0x + \beta_1)$$
$$-(b_0x^2 + b_1x + b_2)(\alpha_0x^2 + \alpha_1x + \alpha_2) = 0,$$

gives the following five equations:

$$\begin{aligned}
&a_0\beta_0 \qquad\qquad - b_0\alpha_0 &&= 0,\\
&a_1\beta_0 - a_0\beta_1 - b_1\alpha_0 - b_0\alpha_1 &&= 0,\\
&a_2\beta_0 - a_1\beta_1 - b_2\alpha_0 - b_1\alpha_1 - b_0\alpha_2 &&= 0,\\
&a_3\beta_0 + a_2\beta_1 \qquad - b_2\alpha_1 - b_1\alpha_2 &&= 0,\\
&\qquad a_3\beta_1 \qquad\qquad - b_2\alpha_2 &&= 0;
\end{aligned}$$

whence

$$R = \begin{vmatrix} a_0 & 0 & -b_0 & 0 & 0 \\ a_1 & -a_0 & -b_1 & -b_0 & 0 \\ a_2 & -a_1 & -b_2 & -b_1 & -b_0 \\ a_3 & a_2 & 0 & -b_2 & -b_1 \\ a & a_3 & 0 & 0 & -b_2 \end{vmatrix}$$

123. Sylvester's Dialytic Method of Elimination. This method leads to the same determinants for resultants as Euler's method; but it is simpler in its application and has an advantage over Euler's method in point of generality, since it can often be applied to form the resultant of equations involving several variables.

To find the resultant of the two equations

$$f(x) \equiv a_0x^m + a_1x^{m-1} + a_2x^{m-2} + \cdots + a_m = 0,$$
$$F(x) \equiv b_0x^n + b_1x^{n-1} + b_2x^{n-2} + \cdots + b_n = 0,$$

of degrees m and n, with one unknown, we multiply the first successively by

$$x^0, x^1, x^2, x^3, \cdots, x^{n-1},$$

and the second by $x^0, x^1, x^2, x^3, \cdots, x^{m-1}$.

We obtain thus the system of equations

$$f(x)=0,\ xf(x)=0,\ x^2f(x)=0, \cdots x^{n-1}f(x)=0,$$

$$F(x)=0,\ xF(x)=0,\ x^2F(x)=0, \cdots x^{m-1}F(x)=0.$$

There are $m+n$ equations, and the highest power of x is

$$m+n-1.$$

If there is a common root, it will satisfy all the equations of this system. And, in taking for unknowns, the different powers of x,

$$x, x^2, x^3, \cdots, x^{m+n-1},$$

the preceding equations form a system of $m+n$ linear equations with $m+n-1$ unknowns.

Hence, by Art. 43, we can eliminate these unknowns and get a resultant, R, which is equal to zero, if the equations are consistent.

EXAMPLES.

1. Find the resultant R of two quadratic equations

$$ax^2+bx+c=0,\ a_1x^2+b_1x+c_1=0.$$

We have

$$\begin{aligned} xf(x) &\equiv ax^3+bx^2+cx &&=0,\\ f(x) &\equiv \phantom{ax^3+{}} ax^2+bx+c &&=0,\\ xF(x) &\equiv a_1x^3+b_1x^2+c_1x &&=0,\\ F(x) &\equiv \phantom{a_1x^3+{}} a_1x^2+b_1x+c_1 &&=0; \end{aligned}$$

from which, eliminating x^3, x^2, x, we get the same determinant as in the preceding article, columns now replacing rows:

$$R \equiv \begin{vmatrix} a & b & c & 0 \\ 0 & a & b & c \\ a_1 & b_1 & c_1 & 0 \\ 0 & a_1 & b_1 & c_1 \end{vmatrix}$$

2. Find the resultant of the two equations

$$f(x) \equiv a_0x^4 + a_1x^3 + a_2x^2 + a_3x + a_4 = 0,$$
$$F(x) \equiv b_0x^2 + b_1x + b_2 = 0.$$

We have the following system:

$$f(x) \equiv 0 \cdot x^5 + a_0x^4 + a_1x^3 + a_2x^2 + a_3x + a_4 = 0,$$
$$xf(x) \equiv a_0x^5 + a_1x^4 + a_2x^3 + a_3x^2 + a_4x + 0 = 0,$$
$$F(x) \equiv 0 \cdot x^5 + 0 \cdot x^4 + 0 \cdot x^3 + b_0x^2 + b_1x + b_2 = 0,$$
$$xF(x) \equiv 0 \cdot x^5 + 0 \cdot x^4 + b_0x^3 + b_1x^2 + b_2x + 0 = 0,$$
$$x^2F(x) \equiv 0 \cdot x^5 + b_0x^4 + b_1x^3 + b_2x^2 + 0 \cdot x + 0 = 0,$$
$$x^3F(x) \equiv b_0x^5 + b_1x^4 + b_2x^3 + 0 \cdot x^2 + 0 \cdot x + 0 = 0.$$

Therefore, we have for the resultant,

$$R = \begin{vmatrix} 0 & a_0 & a_1 & a_2 & a_3 & a_4 \\ a_0 & a_1 & a_2 & a_3 & a_4 & 0 \\ 0 & 0 & 0 & b_0 & b_1 & b_2 \\ 0 & 0 & b_0 & b_1 & b_2 & 0 \\ 0 & b_0 & b_1 & b_2 & 0 & 0 \\ b_0 & b_1 & b_2 & 0 & 0 & 0 \end{vmatrix}$$

124. There are other methods of elimination, notably the method by Symmetric Functions and Bezout's Method, for an explanation of which we refer the student to a higher work on the subject, such as Burnside and Panton's *Theory of Equations.*

We shall close this chapter by giving some examples illustrative of the methods that we have considered in the foregoing articles.

EXAMPLES.

1. Eliminate, by the method of Art. 122, x from the two quadratic equations

$$x^2 + 4x - 21 = 0, \quad x^2 - 13x + 30 = 0,$$

and show that $R = 0$, and thus prove that the equations have a common factor.

2. Apply the same method to find the resultant of the two cubic equations

$$ax^3 + bx^2 + cx + d = 0,$$

$$a'x^3 + b'x^2 + c'x + d' = 0.$$

3. To solve, making use of Euler's method, the equations:

$$\left.\begin{array}{r} 3y^2 + 4xy + 3x^2 - 9y - 15x = 0, \\ y^2 - 2xy + x^2 + 2y - 10x = 0. \end{array}\right\} \quad . \quad . \quad . \quad . \quad (1)$$

Rearranging the terms according to descending powers of x, we have

$$\left.\begin{array}{r} 3x^2 + (4y - 15)x + 3y^2 - 9y = 0, \\ x^2 - (10 + 2y)x + y^2 + 2y = 0. \end{array}\right\} \quad . \quad . \quad . \quad . \quad (2)$$

These are equations of the second degree with respect to x, of which the coefficients a, b, c, a_1, b_1, c_1 (see Art. 122, Ex. (1)) are respectively

$$\begin{array}{lll} 3, & 4y - 15, & 3y^2 - 9y; \\ 1, & -(10 + 2y), & y^2 + 2y. \end{array}$$

Therefore, by substitution in the value of R of Ex. (1), Art. 123, we have

$$\begin{vmatrix} 3 & 0 & 1 & 0 \\ 4y-15 & 3 & -(10+2y) & 1 \\ 3y^2-9y & 4y-15 & y^2+2y & -(10+2y) \\ 0 & 3y^2-9y & 0 & y^2+2y \end{vmatrix} = 0 \quad . \quad (3)$$

or, in developing,

$$4y(y^3+2y^2-9y-18)=0 \quad . \quad . \quad . \quad . \quad (4)$$

The solution of this equation gives for the roots

$$y=0,\ y=3,\ y=-3,\ y=-2.$$

Then, to calculate the corresponding values of x, in this example, we simply eliminate x^2 between the proposed equations, which gives an equation of the first degree in x

$$(3+2y)x-3y=0.$$

Substituting successively the roots obtained for y, we find

$$x=0,\ x=1,\ x=3,\ x=6.$$

The given equations admit, therefore, four common solutions

$$(0,0),\ (3,1),\ (-3,3),\ (-2,6).$$

4. Find the condition that all the roots of the equation

$$x^3+3Hx+G=0$$

shall be real.

Solution. The three roots may be represented by

$$\alpha,\ \beta+\sqrt{r^2},\ \text{and}\ \beta-\sqrt{r^2}.$$

These will all be real when

$$r^2>0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

and the last two will be imaginary when

$$r^2<0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Now we have, Art. 94, (3),

$$\left.\begin{aligned} \alpha + 2\beta &= 0, \\ 2\alpha\beta + \beta^2 - \gamma^2 &= 3H, \\ \alpha\beta^2 - \alpha\gamma^2 &= -G. \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad (3)$$

To eliminate α and β from (3), we substitute the value of α from the first in the second and third, and then multiply the second by β twice, and the third by β once, thus forming the five equations:

$$3\beta^2 + (\gamma^2 + 3H) = 0,$$

$$3\beta^3 + (\gamma^2 + 3H)\beta = 0,$$

$$3\beta^4 + (\gamma^2 + 3H)\beta^2 = 0,$$

$$2\beta^3 - 2\gamma^2\beta - G = 0,$$

$$2\beta^4 - 2\gamma^2\beta^2 - G\beta = 0,$$

whence, the determinant

$$\begin{vmatrix} 0 & 0 & 3 & 0 & (\gamma^2 + 3H) \\ 0 & 3 & 0 & (\gamma^2 + 3H) & 0 \\ 3 & 0 & (\gamma^2 + 3H) & 0 & 0 \\ 0 & 2 & 0 & -2\gamma^2 & -G \\ 2 & 0 & -2\gamma^2 & -G & 0 \end{vmatrix} = 0.$$

This reduces to

$$\gamma_1 = -\frac{27}{4}\left(\frac{G^2 + 4H^3}{(4\gamma^2 + 9H)^2}\right),$$

which, compared with (1), shows that the roots are all real when

$$G^2 + 4H^3 < 0,$$

the required condition. When

$$G^2 + 4H^3 > 0,$$

the two conjugate roots are imaginary. The function $G^2 + 4H^3$ is called the *discriminant* of the cubic

$$x^3 + 3Hx + G = 0.$$

CHAPTER XI.

SOLUTION OF NUMERICAL EQUATIONS.

125. There is an essential difference between the solutions of algebraic and numerical equations. In the former we have a general result expressed in symbolic characters, and it has been proved to be impossible to carry this solution beyond equations of the fourth degree (Art. 53).

But it is possible to solve numerical equations of a much higher degree, and to obtain at least approximate values of the roots accurate enough for all practical purposes.

To this end, we determine the roots separately, and we must first *separate* the roots; for, before attempting the approximation to any individual root, it is generally necessary that it should be situated in a known interval which contains no other real root. In Chapter IX. certain methods of separating the roots of an equation have been explained.

Real roots of numerical equations are either commensurable or incommensurable. Commensurable roots include integers, fractions, and repeating decimals which can be reduced to fractions; incommensurable roots consist of interminable decimals. The roots of the former class can be found exactly, and those of the latter, as we have just intimated, approximated to with any degree of accuracy. In this chapter we shall consider the solution of numerical equations.

126. Theorem. *If the coefficient of the first term of $f(x)$ is unity and all the other coefficients are whole numbers, any commensurable real root of $f(x) = 0$ is a whole number and an exact divisor of p_n.*

For, if possible, let $\frac{a}{b}$, a fraction in its lowest terms, be a root of the equation

$$f(x) \equiv x^n + p_1x^{n-1} + p_2x^{n-2} + \cdots + p_{n-1}x + p_n = 0;$$

we have then

$$\left(\frac{a}{b}\right)^n + p_1\left(\frac{a}{b}\right)^{n-1} + \cdots + p_{n-1}\left(\frac{a}{b}\right) + p_n = 0;$$

from which, multiplying by b^{n-1}, we obtain

$$-\frac{a^n}{b} = p_1a^{n-1} + p_2a^{n-2}b + \cdots + p_{n-1}ab^{n-2} + p_nb^{n-1}.$$

Now, since $\frac{a^n}{b}$ is a fraction in its lowest terms, this equation is impossible, for an integer cannot be equal to a fraction. Hence $\frac{a}{b}$ cannot be a root of the equation. The real roots of the equation, therefore, are either integers or incommensurable quantities.

It is evident, by Art. 94, that any commensurable root is an exact divisor of p_n. Every equation with finite coefficients can be reduced to the form in which the coefficient of the first term is unity, and those of the other terms whole numbers by the method of Art. 103.

127. Knowing that the integral roots of $f(x)$ are factors of p_n, we can often determine them by trial. To do this, we must first find the limits within which the roots lie (Chap. IX). For example, take the equation

$$x^3 - 4x^2 + x + 6 = 0.$$

Here the real roots lie between $+4$ and -2. The possible commensurable roots, being integral factors of 6, are ± 1, $+2$, $+3$, and we easily find that the roots are -1, $+2$, $+3$.

We shall in the next article explain a general method of obtaining the integral roots of an equation whose coefficients are all integers.

128. Newton's Method of Divisors.

Suppose h to be an integral root of the equation

$$a_0x^n + a_1x^{n-1} + \cdots + a_{n-1}x + a_n = 0. \quad . \quad . \quad . \quad (1)$$

Let the quotient, when the polynomial is divided by $x - h$, be

$$b_0x^{n-1} + b_1x^{n-2} + \cdots + b_{n-2}x + b_{n-1},$$

in which b_0, b_1, etc., are all integers.

Proceeding as in Art. 82, we obtain

$$a_0 = b_0,\ a_1 = b_1 - hb_0,\ a_2 = b_2 - hb_1 \cdots$$

$$a_{n-2} = b_{n-2} - hb_{n-3},\ a_{n-1} = b_{n-1} - hb_{n-2},\ a_n = -hb_{n-1}.$$

The last of these equations proves that a_n is divisible by h, the quotient being $-b_{n-1}$. The second last, which is the same as

$$a_{n-1} + \frac{a_n}{h} = -hb_{n-2},$$

proves that the sum of the quotient thus obtained and the second last coefficient is again divisible by h, the quotient being $-b_{n-2}$: and so on. Continuing the process, the last quotient obtained in this way will be $-b_0$, which is equal to $-a_0$.

In this way we can test all the divisors of a_n and see whether they are roots of the equation. They must, at each step of the above process, give integral quotients and a final quotient equal to $-a_0$. As soon as a fractional quotient is met with, the number that we are trying must be rejected, for it cannot be an integral root. This is called Newton's* Method of Divisors.

* Isaac Newton (1642–1727).

129. Application of the Method of Divisors. In applying this method it is convenient, after a manner analogous to Art. 82, to write the series of operations as follows:

$$\begin{array}{cccccc} a_n & a_{n-1} & a_{n-2} \cdots & a_2 & a_1 & a_0 \\ & -b_{n-1} & -b_{n-2} & -b_2 & -b_1 & -b_0 \\ \hline & -hb_{n-2} & -hb_{n-3} & -hb_1 & -hb_0 & 0 \end{array}$$

The first figure in the second line ($-b_{n-1}$) is obtained by dividing a_n by h. This is to be added to a_{n-1} to obtain the first figure in the third line ($-hb_{n-2}$). This is to be divided by h to obtain the second figure in the second line ($-b_{n-2}$); this is to be added to a_{n-2}, and so on. If h be a root, the last figure in the second line thus obtained will be $-a_0$.

When we have proved in this manner that h is a root, the next operation with any divisor may be performed, not on the original coefficients a_n, a_{n-1}, $\cdots$, but on those of the second line with their signs changed, for these are the coefficients of the quotient when the original polynomial is divided by $x-h$.

We need not include the numbers 1 and -1 in the number of trial divisors. It is more convenient to determine beforehand by trial whether either of these numbers is a root.

EXAMPLES.

1. Find the integral roots of the equation

$$x^4+6x^3+x^2-24x-20=0.$$

We observe that all the roots lie between $+3$ and -6. Hence, the following divisors of 20 are possible roots:

$$-5, \quad -4, \quad -2, \quad -1, \quad +1, \quad +2.$$

By trial we find that -1 is a root, and $+1$ is not. We commence with $+2$.

$$\begin{array}{rrrrr} -20 & -24 & +1 & +6 & +1 \\ & -10 & -17 & -8 & -1 \\ \hline & -34 & -16 & -2 & 0 \end{array}$$

Hence 2 is a root.

We next try -2, making use of the coefficients of the second line with the sign changed.

$$\begin{array}{rrrr} 10 & 17 & 8 & 1 \\ & -5 & -6 & -1 \\ \hline & +12 & +2 & 0 \end{array}$$

Hence -2 is a root.

We proceed next with -4. As this does not divide 5, it is not a root, so we try -5.

$$\begin{array}{rrr} 5 & 6 & 1 \\ & -1 & -1 \\ \hline & 5 & 0 \end{array}$$

and -5 is a root.

One step more in the process would show us, as we already know, that -1 is also a root. Hence the roots of the equation are -1, -2, -5, 2.

2. Find the integral roots of the equation

$$x^4 + 11x^3 + 41x^2 + 61x + 30 = 0.$$

It is evident that there is no positive root. By trial we find that the limit of the negative roots is -6. Hence the possible integral roots are

$$-1, \quad -2, \quad -3, \quad -4, \quad -5.$$

We commence with -5.

$$\begin{array}{rrrrr} 30 & 61 & 41 & 11 & 1 \\ & -6 & -11 & -6 & -1 \\ \hline & 55 & 30 & 5 & 0 \end{array}$$

so -5 is a root.

As 4 will not divide 6, -4 is not a root (as we knew in the beginning, for it does not divide 30), so we try -3, and then -2, and lastly -1, as follows:

$$\begin{array}{rrrr} 6 & 11 & 6 & 1 \\ & -2 & -3 & -1 \\ \hline & 9 & 3 & 0 \end{array}$$

Hence, -3 is a root.

$$\begin{array}{rrr} 2 & 3 & 1 \\ & -1 & -1 \\ \hline & 2 & 0 \end{array}$$

Hence, -2 is a root.

$$\begin{array}{rr} 1 & 1 \\ & -1 \\ \hline & 0 \end{array}$$

Hence, -1 is a root, and the roots are all integral.

3. Find the integral roots of

$$x^5 - 4x^4 - 16x^3 + 46x^2 + 63x - 90 = 0$$

By trial we find that $+1$ is a root; we therefore depress the equation by dividing through by $x - 1$, which gives

$$x^4 - 3x^3 - 19x^2 + 27x + 90 = 0.$$

Ans. $1, 3, 5, -2, -3$.

4. Find all the roots of

$$x^4 - 3x^3 - 11x^2 + 19x + 42 = 0.$$

Here limits of the roots are $+4$ and -3; and the possible integral roots are $+3, +2, +1, -1, -2$.

Ans. $+3, -2, 1 + 2\sqrt{2}, 1 - 2\sqrt{2}$.

5. Find all the roots of the equation

$$x^4 + x^3 - 2x^2 + 4x - 24 = 0.$$

6. Find the integral roots of the equation

$$15x^5 - 19x^4 + 6x^3 + 15x^2 - 19x + 6 = 0.$$

7. Find all the roots of the equation

$$x^4 - 2x^3 - 19x^2 + 68x - 60 = 0.$$

The roots lie between -6 and 6. We find that $2, 3, -5$ are roots, and that the factor left after the final division is $x - 2$; hence 2 is a double root, and the polynomial is therefore equivalent to

$$(x-2)^2(x-3)(x+5).$$

130. Determination of Multiple Roots. The Method of Divisors, as shown by Ex. 7 of the last article, determines multiple roots when they are commensurable. In applying the method, when any divisor of a_n, which is found to be a root, is a divisor of the absolute term of the reduced polynomial, it may also be a root of the latter. If it is, it will be a double root of the proposed equation. If it is found to be a root of the next reduced polynomial, it will be a triple root of the proposed equation, and so on. It is often a saving of labor to seek for multiple roots in this way, rather than by the laborious method of the H. C. F. (Art. 98).

EXAMPLES.

Find the commensurable and multiple roots of

1. $2x^3 - 31x^2 + 112x + 64 = 0.$

2. $x^4 - x^3 - 30x^2 - 76x - 56 = 0.$

3. $x^5 - 8x^4 + 22x^3 - 26x^2 + 21x - 18 = 0.$

131. Newton's Method of Approximation. We shall now proceed to the determination of incommensurable roots, giving first Newton's Method.

In any method of approximation, the root that we are seeking is supposed to be separated from all other roots and to be contained within close limits. Let $f(x) = 0$ be the given

equation, and let a be a known number differing by a small quantity (a decimal fraction), h, say, from the root $a+h$. We have then

$$f(a+h) \equiv f(a) + f'(a)h + \frac{f''(a)}{1 \cdot 2}h^2 + \cdots = 0. \quad . \quad (1)$$

In the first approximation, since h is small, we neglect the terms which contain h^2 and higher powers.

Hence (1) becomes

$$f(a) + f'(a)h = 0,$$

which gives, as a first approximation to the root, the value

$$a - \frac{f(a)}{f'(a)}.$$

Representing this root by b, and applying the same process a second time, we have for a second approximation to the root

$$b - \frac{f(b)}{f'(b)}, \text{ and so on.}$$

The oftener this process is repeated, the more accurate is the approximation. In general the approximation is rapid, but this method has been entirely superseded by Horner's Method, which we take up in the next article. To illustrate this method, consider the equation

$$f(x) \equiv x^3 - 4x^2 - 2x + 4 = 0. \quad . \quad . \quad . \quad . \quad (1)$$

We find that the three roots are comprised respectively in the intervals $(-1, -2)$, $(0, 1)$, $(4, 5)$. Let us first calculate the last one. Narrowing the limits, we find that the root is comprised between 4.2 and 4.3.

We find, then,

$$\frac{f(a)}{f'(a)} = \frac{f(4.2)}{f'(4.2)} = \frac{-0.872}{+17.32} = -0.0504.$$

A first approximation is, therefore, 4.2504.
Calling this b, we have

$$\frac{f(b)}{f'(b)}=\frac{f(4.2504)}{f'(4.2504)}=\frac{0.014825}{18.1843}=0.000815.$$

A second approximation is

$$4.2504-0.000815=4.249585,$$

which will be found to be correct to the third decimal place. In like manner the root between 0 and 1 is found to be 0.85363, and that between -1 and -2, to be -1.102775. Here, as the example is given simply to illustrate the method, no pains has been taken to carry the approximation beyond the third decimal place. As the coefficient of x^2 in equation (1) is 4, the 3 roots added together should give 4.

132. Horner's Method of Solving Numerical Equations. By Horner's method both the commensurable and the incommensurable roots can be obtained. The root is evolved figure by figure; first the integral part (if any), then the decimal part till the root terminates if commensurable, or to any number of places if incommensurable. This method is really an extension of the principles of the method of Art. 107, which involves the diminishing of the roots by known quantities. A root which has several figures is obtained by continued applications of that method, the successive transformations being exhibited in a compact form, as will be made apparent by the examples given below.

The first step in the solution of a numerical equation is to find the *first figure* of the root. This can usually be done by trial, though sometimes it may be necessary to resort to one of the methods of Chapter IX to separate the roots.

EXAMPLES.

1. Find the positive roots of the equation

$$8x^3 - 260x^2 - 546x - 207 = 0.$$

There can be only one positive root; and it is found by trial to lie between 30 and 40. Thus the first figure of the root is 3. We now diminish the roots by 30. The transformed equation will have one root between 0 and 10. It is found to lie between 4 and 5. We next diminish the roots of the transformed equation by 4, so that the roots of the proposed equation will be diminished by 34. The second transformed equation will have one root between **0** and **1**. On diminishing the roots of this latter equation by .5, we find that its absolute term is reduced to zero; that is, the diminution of the roots of the proposed equation by 34.5 reduces its absolute term to zero.

Therefore, 34.5 is a root of the given equation. The method of calculation is exhibited as follows:

8	− 260	− 546	− 207	34.5
	240	− 600	− 34380	
	− 20	− 1146	**− 34587**	
	240	6600	29688	
	220	**5454**	**− 4899**	
	240	1968	4899	
	460	7422	**0**	
	32	2096		
	492	**9518**		
	32	280		
	524	9798		
	32			
	556			
	4			
	560			

The broken lines mark the conclusion of each transformation, and the figures in dark type are the coefficients of the successive transformed equations. (See Art. 107.)

Thus $$8x^3 + 460x^2 + 5454x - 34587 = 0$$

is the first transformed equation, whose roots are less by 30 than the roots of the proposed equation, and are found to lie between 4 and 5. And

$$8x^3 + 556x^2 + 9518x - 4899 = 0$$

is the second transformed equation.

If this second transformed equation had not an exact root .5, we should find the limits between which the root lies, and then proceed as before, and so on.

2. Find the positive root of the equation

$$4x^3 - 13x^2 - 31x - 275 = 0. \quad . \quad . \quad . \quad . \quad (1)$$

Here the arithmetical calculation is as follows:

4	−13	−31	−275	6.25
	24	66	210	
	11	35	**−65**	
	24	210	51.392	
	35	**245**	**−13.608**	
	24	11.96	13.608	
	59	256.96	0	
	.8	12.12		
	59.8	**269.08**		
	.8	3.08		
	60.6	272.16		
	.8			
	61.4			
	.2			
	61.6			

We find by trial that the proposed equation has its positive root between 6 and 7. The first figure of the root is, therefore, 6.

Diminish the roots by 6. The transformed equation

$$4x^3 + 59x^2 + 245x - 65 = 0$$

has a root between 0 and 1. It is found by trial to lie between .2 and .3.

Diminish the roots again by .2. The transformed equation

$$4x^3 + 61.4x^2 + 269.08x - 13.608 = 0$$

is found to have the root .05. Hence 6.25 is a root of the proposed equation.

It is convenient in practice to avoid the use of the decimal points. This can easily be effected as follows:

When the decimal part of the root (suppose $.abc \cdots$) is about to appear, multiply the roots of the corresponding transformed equation by 10; that is, annex one zero to the right of the figure in the first column, two to the right of the figure in the second column, three to the right of that in the third; and so on, if there be more columns (Art. 103). The root of the transformed equation is then, not $.abc \cdots$, but $a.bc \cdots$.

Diminish the roots by a. The transformed equation has a root $.bc \cdots$. Multiply the roots of this equation again by 10. The root becomes $b.c \cdots$, and the process is continued as before.

To illustrate this we repeat the above operation, omitting the decimal points. In subsequent examples in this book this simplification will be adopted, and the student is advised to make use of this principle in the solution of all such examples.

4	−13	−31	−275	6.25
	24	66	210	
	11	35	−65000	
	24	210	51392	
	35	24500	−13608000	
	24	1196	13608000	
	590	25696	0	
	8	1212		
	598	2690800		
	8	30800		
	606	2721600		
	8			
	6140			
	20			
	6160			

In the examples here considered the root terminates at an early stage. When there are many more figures in the root, the process would become very laborious, if it were not for a simplification which we shall explain in the next article. This introduces to us one of the most valuable practical advantages of Horner's Method, which is, that after the second or third (sometimes even after the first) figure of the root is found, the *transformed equation itself suggests, by mere inspection, the next figure of the root.*

133. Principle of the Trial-divisor. We have seen in Art. 131 that when an equation is transformed by the substitution of $a+h$ for a, a being a number differing from the true root by a quantity h, small in proportion to a, an approximate numerical value of h is $\dfrac{f(a)}{f'(a)}$.

Now, as in the successive *transformed equations* of Horner's method, the last coefficient is $f(a)$ and the next to the last is $f'(a)$, we would evidently get the next figure of the root by dividing $f(a)$ by $f'(a)$; that is, by dividing the last coefficient by the coefficient next to the last. This will, in general, give the correct figure only after two or three steps in the process have been completed, and the part of the root to be found bears a small ratio to the part already evolved. We might, therefore, if we pleased, at any stage of Horner's operations, apply Newton's method to get a further approximation to the root. The second last coefficient of each transformed equation is called the *trial-divisor*. It is evident that the application of this principle will greatly facilitate the work, but we must use due care not to apply Newton's method too soon.

Thus, in the second example of the last article, the number **5** is correctly suggested by the trial-divisor 2690800, for 2690800 into 13608000 goes **5** times (and something over, of course). In this example, indeed, the second figure of the root is correctly suggested by the trial-divisor of the first transformed equation; although, in general, such is not the case. In practice the student must estimate the probable effect of the leading coefficients of the transformed equation. To illustrate, consider the following examples:

EXAMPLES.

1. Find the roots of the equation $x^3 - 7x + 7 = 0$.

We first separate the roots by Sturm's Theorem (Ex. 4, Art. 119). We find that there are two positive roots between **1** and **2**, and a negative root between -3 and -4. Transforming the equation by diminishing the roots by **1**, we find that of the two positive roots, one lies between **1.3** and **1.4**, and the other between 1.6 and 1.7.

We shall find the first root to five decimal places, and leave it as an exercise for the student to find the root between 1.6 and 1.7 and the negative root.

The calculation is written as follows:

1 0	−7	+7	1.35689
1	1	−6	
1	−6	1000	
1	2	−903	
2	−400	97000	
1	99	− 86625	
30	−301	10375000	
3	108	−9048984	
33	−19300	1326016000	
3	1975	−1184429568	
36	−17325	141586432000	
3	2000		
390	−1532500		
5	24336		
395	−1508164		
5	24372		
400	−148379200		
5	325504		
4050	−148053696		
6	325568		
4056	−14772812800		
6			
4062			
6			
40680			
8			
40688			
8			
40696			
8			
407040			

Here we first diminish the roots by 1. As the decimal part is about to appear, attach ciphers to the coefficients of the transformed equation, which thus becomes

$$x^3 + 30x^2 - 400x + 1000 = 0.$$

We next diminish the roots by 3, as we have already found that 3 is the next figure of the root sought. After multiplying the roots by 10, the second transformed equation is

$$x^3 + 390x^2 - 19300x + 97000 = 0.$$

The trial divisor now becomes effective; 19300 into 97000 goes 5 times, and 5 is found to be the next figure of the root. If we had adopted the figure 6, the absolute term would have become negative, the change of sign showing that we had gone beyond the root. We must take care that, at least after the first transformation, the absolute term preserves its sign throughout the operation. The figure to be adopted in every case as part of the root is *that highest number which in the process of transformation will not change the sign of the absolute term.* If we were to take by mistake a number too small, the error would show itself, just as in ordinary division or evolution, by the next suggested number being greater than 9.

After diminishing by 5 the roots of the second transformed equation (and multiplying the roots of the resulting equation by 10), the next figure of the root is 6, for 1532500 goes into 10375000 6 times. And so we proceed, as indicated in the above operation. Of course the process can be continued indefinitely, and the root obtained correct to any number of decimal places.

2. Find, to 5 decimal places, the positive root of the equation

$$x^4 - 8x^3 + 14x^2 + 4x - 8 = 0,$$

which lies between 2 and 3.

3. Find the two positive roots of the equation

$$x^4 + 4x^3 - 4x^2 - 11x + 4 = 0.$$

There are several abbreviations of Horner's process, by which, after three or four places of decimals have been calculated as above, several more may be correctly obtained by a contracted process, for an explanation of which we refer the reader to Burnside and Panton's *Theory of Equations.*

134. Negative Roots. To obtain a negative root of $f(x)=0$, we simply form the equation $f(-x)=0$ (Art. 102), and get its corresponding positive root, which will be the required negative root of $f(x)=0$.

EXAMPLES.

1. Find the negative root of the equation

$$x^4-12x^2+12x-3=0.$$

2. Find the root between 3 and 4 of the equation

$$x^4-10x^2-4x+8=0,$$

to four places of decimals.

3. Find the real roots of the equation

$$x^4-12x+7=0. \quad \textit{Ans. } 2.0473;\ .5037.$$

MISCELLANEOUS EXAMPLES.

Find the quotient and remainder when

1. $x^6-2x^5+3x^4-6x^3-10x+6$ is divided by $x-2$.

2. $x^4+3x^3-2x^2+x-4$ is divided by $x-3$.

3. $4x^3+2x^2+5x-9$ is divided by $x+4$.

4. $2x^5+x^4-4x^3+8x^2-2x+16$ is divided by $x+6$.

5. $x^{10}+2x^9-4x^7-5x^6+x^5-x^4+10$ is divided by $x-5$.

6. Trace the polynomial

$$4x^3 - x - 8.$$

7. Solve the equation

$$x^3 + 288x + 1216 = 0,$$

which has a root $2 - 10\sqrt{-3}$.

8. Form a rational sextic equation which shall have for three of its roots

$$1 - 3\sqrt{2}, \quad 2 + \sqrt{-1}, \quad 3 - 2\sqrt{-1}.$$

9. Solve the cubic

$$x^3 + 100x^2 + 100x + 1000,$$

one root being $\sqrt{-10}$.

Find by Descartes' rule an inferior limit to the number of imaginary roots of the following equations:

10. $x^9 - 3x^7 + x^5 - x^4 - x + 6 = 0.$

11. $x^7 + 4x^6 + x^5 + 3x^4 + x^3 + x^2 + x + 1 = 0.$

12. $x^{10} + 2x^8 - 5x^5 - x^4 + x^3 + 4x^2 - 6 = 0.$

13. $x^8 - 5x^4 + 3 = 0.$

14. $4x^3 + 7x^2 - 18x - 30 = 0.$

15. Solve the equation

$$27x^3 + 42x^2 - 28x - 8 = 0,$$

whose roots are in geometric progression. [Art. 94.]

Ans. $-2, \frac{2}{3}, -\frac{2}{9}$.

16. The equation

$$x^4 - 2x^3 + 4x^2 + 6x - 21 = 0,$$

has two roots equal in magnitude and opposite in sign; determine all the roots. Take $\alpha + \beta = 0$, and use Art. 94.

Ans. $\sqrt{3}, \sqrt{-3}, 1 \pm \sqrt{-6}$.

17. One of the roots of the cubic

$$x^3 - px^2 + qx - r = 0$$

is double another; show that it may be found from a quadratic equation.

18. Find the condition which must be satisfied by the coefficients of the equation

$$x^3 - px^2 + qx - r = 0,$$

when two of its roots α, β, are connected by a relation $\alpha + \beta = 0$.

Ans. $pq - r = 0$.

19. The product of two unequal roots of the equation

$$ax^3 + bx^2 + cx + d = 0$$

is 1; prove that the third root is $\frac{a-c}{b-d}$.

Solve the following five equations, each of which has equal roots:

20. $x^3 - 5x^2 - 8x + 48 = 0$.

21. $x^4 - \frac{1}{2}x + \frac{3}{16} = 0$.

22. $x^4 - 2x^3 - x^2 - 4x + 12 = 0$.

23. $x^4 + 2x^3 - 12x^2 - 18x + 27 = 0$.

24. $x^7 - 7x^6 + 10x^5 + 22x^4 - 43x^3 - 35x^2 + 48x + 36 = 0$.

Ans. $(x-2)^2(x-3)^2(x+1)^3$.

25. Find the equation whose roots are the roots of

$$x^7 - 6x^6 - x^5 + 2x^3 - x + 7 = 0$$

with their signs changed.

26. Change the equation

$$2x^5 - 4x^3 + 5x^2 - 7x + 3 = 0$$

into another, the coefficient of whose highest term will be unity, and the coefficients of the other terms integers.

Remove the fractional coefficients from the equations:

27. $x^4 - \frac{1}{3}x^3 + \frac{1}{2}x^2 - 5x + 2 = 0.$

28. $x^3 + \frac{2}{11}x^2 - \frac{3}{22} = 0.$

29. $x^3 + \frac{1}{5}x^2 - \frac{1}{2}x + 3 = 0.$

30. Find the equation whose roots are the reciprocals of the roots of

$$x^3 - 91x^2 - 910x + 1000 = 0.$$

31. Give condition that the following equation should have, (1) one infinite root, (2) two infinite roots.

$$(a^2 - 4)x^4 + (c - 7)x^3 + ax^2 - cx + 20 = 0.$$

32. Increase by 5 the roots of the equation

$$2x^4 - x^3 + 6x^2 + 3x - 10 = 0.$$

33. Increase by 3 the roots of the equation

$$x^3 - 3x^2 + x - 7 = 0.$$

34. Diminish by 2 the roots of the equation

$$x^6 - x^4 + x^2 - x + 5 = 0.$$

35. Diminish by 6 the roots of the equation

$$x^3 - 3x^2 - 2x + 16 = 0.$$

36. Diminish by 1 the roots of the equation

$$x^7 - 2x^6 + x^5 + 4x^3 - x^4 + 7x^2 - 16 = 0.$$

37. Increase by 10 the roots of the equation

$$3x^4 - 6x^3 - 4x^2 + 2x - 5 = 0.$$

Transform each of the following three equations into another wanting the second term:

38. $x^3 - 3x^2 + 4x - 4 = 0.$

39. $x^4 - 8x^3 + 5 = 0.$

40. $2x^3 + 12x^2 - 3x + 5 = 0.$

41. Remove the third term in the equation

$$x^4 - 8x^3 + 18x^2 - 15x + 14 = 0.$$

Remove the second term and solve the two cubic equations (Art. 110):

42. $x^3 - 18x^2 + 157x - 510 = 0.$ *Ans.* $6,\ 6 \pm 7\sqrt{-1}.$

43. $x^3 - 7x^2 + 14x = 20.$ *Ans.* $5,\ 1 + \sqrt{-3},\ 1 - \sqrt{-3}.$

Find a superior limit to the positive and negative roots of the equations:

44. $x^4 - 5x^3 + 37x^2 - 3x + 39 = 0.$

45. $x^5 + 7x^4 - 12x^3 - 49x^2 + 52x - 13 = 0.$

Apply Sturm's theorem to determine the number and situation of the real roots of the following five equations:

46. $x^4 - 4x^3 + 7x^2 - 6x - 4 = 0.$

47. $x^4 - 5x^3 + 10x^2 - 6x - 21 = 0.$

48. $x^5 - 10x^3 + 6x + 1 = 0.$

Ans. Roots all real; one in the interval $\{-4, -3\}$; two in the interval $\{-1, 0\}$; and positive roots in the intervals $\{0, 1\}$, $\{3, 4\}$.

49. $x^4 - 2x^2 - 4x + 10 = 0.$

50. $x^3 - 4x^2 - 4x + 20 = 0.$

51. Show, by Sturm's theorem, that all the roots of the equation

$$x^4 + 3x^3 - x^2 - 3x + 11 = 0$$

are imaginary.

Find the integral roots of the following equations:

52. $x^4 - 5x^3 + 25x - 21 = 0.$

53. $9x^6 + 30x^5 + 22x^4 + 10x^3 + 17x^2 - 20x + 4 = 0.$

54. $x^4 + 6x^3 - 22x^2 - 33x + 54 = 0.$

55. Find the commensurable and multiple roots of

$$x^4 + 12x^3 + 32x^2 - 24x + 4 = 0.$$

Ans. The equation has two pairs of equal roots, both incommensurable.

56. Find the commensurable and multiple roots of

$$x^6 - 8x^5 + 20x^4 - 32x^3 + 68x^2 - 32x + 64 = 0.$$

Ans. $(x-4)^2(x^2+2)^2 = 0.$

57. Find, by Horner's method, to six decimal places, the root between 2 and 3 of the equation

$$x^3 - 49x^2 + 658x - 1379 = 0.$$

58. Find the two real roots of the equation

$x^4 - 11727x + 40385 = 0.$ *Ans.* 3.45592, 21.43067.

Find all the roots of the three equations:

59. $x^3 + x^2 - 2x - 1 = 0.$ *Ans.* -1.80194, -0.44504, 1.24698.

60. $x^3 - 315x^2 - 19684x + 2977260 = 0.$

61. $x^5 - 10x^3 + 6x + 1 = 0.$

$$Ans. \begin{cases} -3.0653315791, \\ -0.6915762280, \\ -0.1756747799, \\ +0.8795087708, \\ +3.0530581162. \end{cases}$$

APPENDIX A.

The definitions of algebraic and transcendental functions given in Art. 50, page 78, are somewhat broader than those found in our elementary text-books on Algebra. That these definitions are exact and cover the entire ground, is evident from the following considerations:

In mathematics there are only four fundamental operations, namely: addition, subtraction, multiplication, and division. If two quantities, x and y, are so related that when one of them is given the other can be calculated, the one is said to be a mathematical *function* of the other. Mathematical functions are further divided into two great classes according as the number of fundamental operations is finite or infinite, in order to calculate the function when the other quantity is given.

If the number of such operations is finite, the function is said to be *algebraic;* otherwise, *transcendental*. For example; in

$$3x^2+2x+1,\ \frac{3x+2}{x^2-5},\ (2x-5)^{12},\ \frac{1}{2x^2+3x+1},$$

the number of operations on x is *finite*. Such expressions are algebraic functions.

The student familiar with trigonometry will recall that $\sin x$, e^x, $\log(1+x)$, as functions of x, are

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots, \text{ without end,}$$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots, \text{ without end,}$$

$$\log(1+x) = x - \tfrac{1}{2}x^2 + \tfrac{1}{3}x^3 - \cdots, \text{ without end.}$$

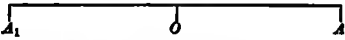
These are examples of transcendental functions, for the number of operations is infinite.

It is also customary to consider expressions like $\sqrt{x^2+2x}$, $\sqrt[3]{x^2+x+1}$ as algebraic functions, although to derive their *true* value, for values of x other than special ones, would imply an infinite number of operations. This seeming inconsistency may be explained on the ground that the extraction of the root, though involving, possibly, an infinite number of the four fundamental operations, is counted as a single (though complex) operation, — making the total number of operations finite (in thought).

APPENDIX B.

Argand's Diagram: In Article 67 and the foot-note, it is possible that rather too much credit is given to Argand.

Kossak * says that Kuhn, in *Novi Commentarii Acad. Petrop. III, ad 1750–1751*, was the first to give geometric expression to $\sqrt{-1}$. Thus lay off $OA_2 = 1$,



$OA_1 = -1$, and draw the perpendicular OB to meet semicircle on A_1A_2 at B. Then $\overline{OB}^2 = OA_1 \cdot OA_2$,

or $$\overline{OB}^2 = -1. \quad \therefore\ OB = \sqrt{-1}. \dagger$$

* *Elemente der Arithmetik.*
† See also Cajori's *History of Mathematics*, page 317.